2009

58th Czech and Slovak Mathematical Olympiad

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First Round of the 58th Czech and Slovak Mathematical Olympiad
Problems for the take-home part
(October 2008)

1. *Find all real solutions of the system*

$$2\sin x\cos(x+y)+\sin y=1,$$
$$2\sin y\cos(y+x)+\sin x=1.$$

Solution. Using the familiar formulas

$$\cos(x+y)=\cos x\cos y-\sin x\sin y,\quad \sin 2x=2\sin x\cos x,\quad \cos 2x=1-2\sin^2 x$$

we can rewrite the left-hand side of the first equation as

$$\begin{aligned}2\sin x\cos(x+y)+\sin y&=2\sin x(\cos x\cos y-\sin x\sin y)+\sin y=\\&=2\sin x\cos x\cos y+(1-2\sin^2 x)\sin y=\\&=\sin 2x\cos y+\cos 2x\sin y=\\&=\sin(2x+y).\end{aligned}$$

Similarly the left-hand side of the second equation equals $\sin(2y+x)$. The given system is thus equivalent to

$$\begin{aligned}\sin(2x+y)&=1,\\ \sin(2y+x)&=1.\end{aligned}\tag{1}$$

Since the sine function assumes the value 1 precisely at points of the form $\frac{1}{2}\pi+2k\pi$, where k is an integer, the solutions of the last system will be precisely those pairs (x,y) for which there exist integers k, l such that

$$2x+y=\tfrac{1}{2}\pi+2k\pi,\quad 2y+x=\tfrac{1}{2}\pi+2l\pi.\tag{2}$$

Multiplying the first equation by two and subtracting the second (or expressing one of the variables from the first equation and substituting it into the second), we obtain after a small manipulation

$$x=\tfrac{1}{6}\pi+\tfrac{2}{3}(2k-l)\pi,\quad y=\tfrac{1}{6}\pi+\tfrac{2}{3}(2l-k)\pi.$$

The solutions of the system are therefore the pairs $(\frac{1}{6}\pi+\frac{2}{3}(2k-l)\pi, \frac{1}{6}\pi+\frac{2}{3}(2l-k))$, where k, l are arbitrary integers. Verification is not necessary, since it is clear from the argument that these pairs satisfy the relations (2), and, hence, also the system (1), which is equivalent to the original system.

Remark. The result can be written also in a different way: since $y - x = \frac{1}{3}(6l - 6k)\pi = 2(l-k)\pi$, setting $m = l - k$, $n = 2k - l$ we can write $x = \frac{1}{6}\pi + \frac{2}{3}n\pi$, $y = x + 2m\pi$, so the solutions are the pairs $(\frac{1}{6}\pi + \frac{2}{3}n\pi, \frac{1}{6}\pi + \frac{2}{3}n\pi + 2m\pi)$, where m, n are arbitrary integers. (If k, l range over all possible pairs of integers, then so do m, n.)

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2. *A cyclic quadrangle $ABCD$ is given. Show that the line connecting the orthocenter of the triangle ABC with the orthocenter of the triangle ABD is parallel to the line CD.*

Solution. Denote by k the circumcircle of the quadrangle $ABCD$, and by U and V the orthocenters of the triangles ABC and ABD, respectively (Fig. 1).

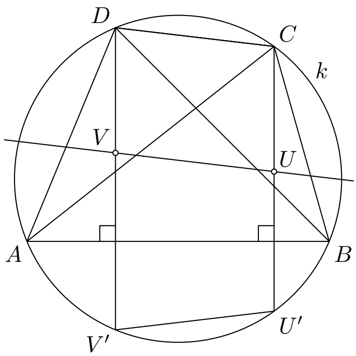

Fig. 1

The image U' of the point U under the symmetry with respect to the side AB lies on the circumcircle of the triangle ABC, i.e. on k. (This holds even if the triangle ABC is obtuse.) Similarly, the image V' of the point V under the same symmetry lies on k.

Assume that the triangles ABC and ABD are acute. The points U, V then lie in the half-plane ABC. The two perpendiculars CU' and DV' to the side AB are parallel, thus the quadrangle $CU'V'D$ is a cyclic trapezoid, which has to be equilateral. From this fact and properties of axial symmetry we obtain the equalities

$$|\angle CDV'| = |\angle U'V'D| = |\angle UVV'|.$$

Since the points C and U lie in the same half-plane determined by the line $V'D$, the lines CD and UV are parallel, which is what we wanted to prove. (We are using the fact that the points D, V, and V' lie on the line in this order.)

In the case when at least one of the triangles ABC and ABD is obtuse, the argument is quite analogous. The points C, D, V' and U' always form an equilateral trapezoid, though not necessarily with vertices in this order.

3. *Find all pairs of natural numbers x, y such that $\frac{xy^2}{x+y}$ is a prime.*

Solution. Assume that the natural numbers x, y and a prime p satisfy the equation

$$\frac{xy^2}{x+y} = p. \tag{1}$$

Denote by d the greatest common divisor of x and y. Then $x = da$ and $y = db$, where a and b are relatively prime. Substituting into (1), clearing the denominators and dividing by the positive number d, we obtain

$$d^2ab^2 = p(a+b). \tag{2}$$

Since the numbers a and b are relatively prime, so are the two numbers b^2 and $a+b$ from the two sides of the equality (2). It follows[1] that $b^2 \mid p$. The prime p has only two divisors: 1 and p, of which the second is not a square; thus necessarily $b = 1$. Substituting this into (2) gives

$$d^2a = p(a+1). \tag{3}$$

Arguing again as above, i.e. since a is a divisor of the left-hand side of (3), it is also a divisor of the right-hand side, and as a, $a+1$ are relatively prime, we conclude that $a \mid p$. Thus either $a = 1$, or $a = p$. We discuss these two cases separately.

If $a = 1$, (3) reads $d^2 = 2p$. As p is a prime, $2p$ can be a square only if $p = 2$. Then also $d = 2$ and we arrive at the first solution $x = da = 2$, $y = db = 2$.

If $a = p$, then dividing by the positive number p (3) becomes $d^2 = p+1$, or $p = (d+1)(d-1)$. As p is a prime and $d-1 < d+1$, necessarily $d-1 = 1$ and $d+1 = p$. This gives $d = 2$, $p = 3$, and we have arrived at the second solution $x = da = dp = 6$ and $y = db = 2$.

It is easy, though not necessary (all the manipulations we performed were equivalent), to check that both pairs are indeed solutions of the original equation (1).

Answer: there are exactly two such pairs (x, y), namely, $(2, 2)$ and $(6, 2)$.

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4. *Consider the infinite arithmetic sequence*

$$a, a+d, a+2d, \ldots, \tag{$*$}$$

where a, d are natural numbers (i.e. positive integers).

- *a) Find an example of the sequence ($*$) which contains infinitely many k-th powers of natural numbers, for all $k = 2, 3, \ldots$.*
- *b) Find an example of the sequence ($*$), which does not contain a k-th power of a natural number, for all $k = 2, 3, \ldots$*
- *c) Find an example of the sequence ($*$), which does not contain a square of a natural number, but contains infinitely many cubes of natural numbers.*

[1] If k and l are relatively prime and $k \mid lm$, then $k \mid m$.

d) *Show that for all natural numbers a, d, k ($k > 1$), the following assertion is true: The sequence $(*)$ either does not contain a k-th power of a natural number, or contains infinitely many k-th powers of natural numbers.*

Solution. a) Set, for instance, $a = 1$, $d = 1$. The sequence $(*)$ then has the form

$$1, 2, 3, 4, \dots,$$

i.e. contains all natural numbers; among these, there are clearly infinitely many k-th powers for any k.

b) Set, for instance, $a = 2$, $d = 4$. The sequence $(*)$ then has the form

$$2, 6, 10, 14, \dots,$$

i.e. is formed by all even numbers of the form $4n + 2$, where $n = 0, 1, 2, \dots$. This sequence certainly does not contain a k-th power of an *odd* number. On the other hand, a k-th power of an arbitrary *even* number is divisible by 2^k, hence also by 4 (since we are considering only $k \geqslant 2$); but no number of the form $4n + 2$ is divisible by 4. Our sequence therefore does not contain a k-th power of any natural number, no matter which $k = 2, 3, \dots$ we choose.

c) Set, for instance, $a = 8$, $d = 16$. The sequence $(*)$ then has the form

$$8, 24, 40, 56, \dots,$$

i.e. consists precisely of the odd multiples $8(2n + 1)$ of eight, where $n = 0, 1, 2, \dots$. This sequence cannot contain a square, because the prime factorization of $8(2n + 1)$ contains the prime 2 with multiplicity three ($8 = 2^3$ and $2n + 1$ is odd), whereas in the prime factorization of a square, each prime occurs with an *even* multiplicity.

On the other hand, the sequence contains the infinitely many cubes $8 \cdot 1^3, 8 \cdot 3^3, 8 \cdot 5^3, \dots$, since the cube of an odd number is again odd, thus of the form $2n + 1$, and our sequence consists of all numbers of the form $8(2n + 1)$.

d) Assume that for the given $k > 1$, the sequence $(*)$ contains at least one k-th power, say, the number m^k for some natural m. Thus $m^k = a + nd$ for some nonnegative integer n. We claim that the sequence $(*)$ then contains also all the (infinitely many) powers $(m + d)^k$, $(m + 2d)^k$, $(m + 3d)^k$, $\dots$.

Indeed, if t is any positive integer, then by the binomial theorem

$$\begin{aligned}(m + td)^k &= m^k + km^{k-1}td + \binom{k}{2}m^{k-2}t^2d^2 + \dots + kmt^{k-1}d^{k-1} + t^kd^k = \\ &= m^k + d \cdot \left(km^{k-1}t + \binom{k}{2}m^{k-2}t^2d + \dots + kmt^{k-1}d^{k-2} + t^kd^{k-1}\right) = \\ &= m^k + d \cdot M = (a + nd) + dM = a + d(n + M).\end{aligned}$$

Since M (the expression in the big parentheses) is clearly a positive integer, the number $(m + td)^k = a + d(n + M)$ is a member of our sequence $(*)$, and the claim follows.

5. *In each vertex of a regular 2008-gon there lies one coin. We choose two coins and move each of them into an adjacent vertex, one in the clockwise direction, the other anti-clockwise. Decide if, continuing in this fashion, it is possible to move all the coins into*

a) 8 heaps of 251 coins each,

b) 251 heaps of 8 coins each.

Solution. Enumerate the vertices of the given polygon successively by the numbers 1, 2, ... , 2 008.

a) We describe a way of moving the coins so as to obtain eight 251-coin heaps.

First of all, move successively the coins from the first 251 vertices $1, 2, \ldots, 251$ into a single heap in vertex 251, compensating their movements by moving "symmetrically" the coins from the last 251 vertices $1\,758, 1\,759, \ldots, 2\,008$ into the vertex $1\,758$. This produces the first two 251-coin heaps. We continue by moving analogously all the coins from the vertices 252 to 502 into a single heap in vertex 502, again compensating by symmetrical movements leading to another 251-coin heap in the vertex with number $1757 - 250$, i.e. 1507. We perform this procedure two more times, in the end obtaining the last two 251-coin heaps in the two adjacent vertices 1004 and 1005.

b) We show that this cannot be achieved.

Assign to each coin the number of the vertex in which it lies (at the moment). Let us see how the sum S of all these 2 008 numbers changes upon moving any pair of coins in the manner allowed. If none of the coins is moved between the vertices 1 and 2 008, the value of S is clearly unchanged, since for one of the coins its assigned number increases by 1, while for the other coin being moved it decreases by 1 (while for all the remaining coins, which are not being moved, the assigned numbers do not change). Similarly, S does not change if we simply move a coin from 2008 into 1 and the other coin from 1 into 2008. Finally, if one of the coins is moved between 1 and 2008 and the other between some other pair of vertices, then S changes into $S \pm 2\,008$, since the numbers assigned to the two coins that are being moved will either both increase, or both decrease, by the values of 1 and 2 007, respectively (in both cases).

Summarizing, we thus see that from its original value $S_0 = 1 + 2 + \cdots + 2\,008 = 1\,004 \cdot 2\,009$, the sum S can only assume values of the form $S = S_0 + 2\,008k$, where k is an integer.

If it were possible to move all the coins into 251 eight-coin heaps, say, at the vertices with numbers $v_1, v_2, \ldots, v_{251}$, then we would have the equality

$$1\,004 \cdot 2\,009 + 2\,008k = 8(v_1 + v_2 + \cdots + v_{251}),$$

which cannot hold for any integer k, since the right-hand side is a multiple of eight, while the left-hand side is not (the number $2\,008k$ is divisible by eight, but $1\,004 \cdot 2\,009$ is not). Consequently, no such sequence of movements of the coins exists.

6. *A triangle ABC is given. In the interior of its sides AC, BC, there are given points E, D, respectively, such that $|AE| = |BD|$. Denote by M the midpoint of the side AB, and by P the intersection of the lines AD and BE. Show that the image of the point P in the central symmetry with respect to M lies on the bisector of the angle ACB.*

Solution. Denote by Q the image of P in the central symmetry with center M. The point Q will lie on the bisector of the angle ACB if and only if it has the same distance from the lines AC and BC. Since the segments AE and BD have the same length, we see that Q will have the same distance from the lines AC and BC if and only if the triangles AEQ and BDQ have the same area (Fig. 2). We now prove the equality of these areas.

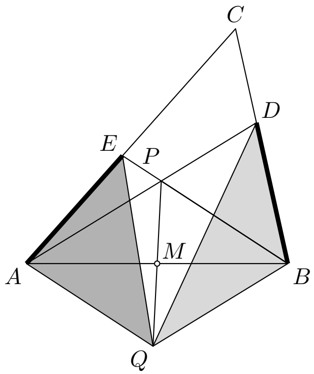

Fig. 2

From the construction of the point Q it follows that $AQBP$ is a parallelogram, i.e. the line QB is parallel to AD; the triangles QBD and QBA thus have equal areas (having equal altitudes to the common base QB). Similarly from $QA \parallel BE$ it follows that the areas of the triangles QAE and QAB are equal. Thus the areas of the triangles AEQ and BDQ are equal as well, as claimed.

First Round of the 58th Czech and Slovak Mathematical Olympiad (December 2nd, 2008)

1. *Find all pairs of positive integers m and n for which*

$$\sqrt{m^2-4} < 2\sqrt{n} - m < \sqrt{m^2-2}.$$

Solution. If m, n satisfy the given inequalities, then clearly

$$m \geqslant 2 \quad \text{and} \quad 2\sqrt{n} - m > 0, \tag{1}$$

otherwise the square roots would not be defined or the middle term $2\sqrt{n}-m$ would not be positive, and thus could not be greater than the nonnegative expression $\sqrt{m^2-4}$.

Assuming that the conditions (1) are satisfied, we can make the following *equivalent* manipulations of the given inequalities (note that in each of the four squarings both sides are defined and nonnegative, and equally both divisions are by the *positive* number n, so all these manipulations are correct):

$$\begin{array}{rl|l@{\qquad}rl|l}
2\sqrt{n} - m &< \sqrt{m^2-2} & ^2 & \sqrt{m^2-4} &< 2\sqrt{n} - m & ^2 \\
4n - 4m\sqrt{n} + m^2 &< m^2 - 2 & & m^2 - 4 &< 4n - 4m\sqrt{n} + m^2 & \\
n + \frac{1}{2} &< m\sqrt{n} & ^2 & m\sqrt{n} &< n + 1 & ^2 \\
n^2 + n + \frac{1}{4} &< m^2 n & :n & m^2 n &< n^2 + 2n + 1 & :n \\
n + 1 + \dfrac{1}{4n} &< m^2 & & m^2 &< n + 2 + \dfrac{1}{n} &
\end{array}$$

The last two inequalities hold if and only if the number m^2 lies in the open interval

$$\left(n + 1 + \frac{1}{4n}, n + 2 + \frac{1}{n}\right).$$

In view of the obvious inequalities $0 < \frac{1}{4n} \leqslant \frac{1}{4}$ and $0 < \frac{1}{n} \leqslant 1$, this interval contains the single integer $n+2$. Under the hypothesis (1), the natural numbers m and n thus satisfy the original inequalities if and only if $m^2 = n + 2$.

It remains to find which positive integers m, n related by $n = m^2 - 2$ satisfy under the hypothesis $m \geqslant 2$ also the second of the conditions (1). We perform the following equivalent manipulations:

$$\begin{aligned}
2\sqrt{m^2-2} - m &> 0, \\
2\sqrt{m^2-2} &> m, \quad \Big|^2 \\
4(m^2-2) &> m^2, \\
3m^2 &> 8.
\end{aligned}$$

The last inequality holds, however, for any $m \geqslant 2$.

Answer. The sought pairs are precisely those of the form $(m, n) = (m, m^2 - 2)$, where $m \geqslant 2$ is any positive integer.

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2. *Let ABC be an acute triangle whose interior angle at the vertex A has magnitude $45°$. Denote by D the foot of the perpendicular from the vertex C, and let P be an arbitrary interior point of the altitude CD. Show that the lines AP and BC are perpendicular if and only if the segments AP and BC are congruent.*

Solution. We start with the first implication. Let $AP \perp BC$; then the point P is the orthocenter of the triangle ABC. We need to show that the segments AP and BC are congruent; we do this by finding two congruent triangles, in which these segments are a pair of corresponding sides.

Denote by E the intersection of the line BP with the side AC, i.e. E is the foot of the altitude from the vertex B. From the right triangle ABE and the given magnitude of the angle BAC, it is easy to see that $|\angle PBD| = 45°$. The triangle PDB is thus right and isosceles, hence $|DP| = |DB|$ (Fig. 1). The similar triangle ADC is right and, in view of the magnitude of the angle at the vertex A, also isosceles; thus $|DA| = |DC|$. By the *sas* theorem, the right triangles APD and CBD are congruent, and their hypotenuses AP, BC thus have equal lengths.

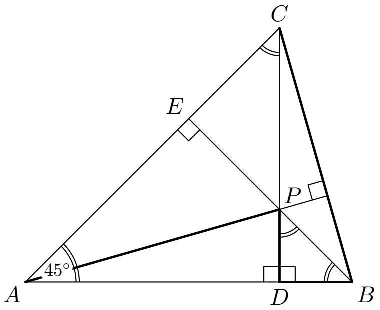

Fig. 1

It remains to show the converse implication. Assume that $|AP| = |BC|$. Since ADC is an isosceles right triangle, necessarily $|AD| = |CD|$, so the triangles PAD and BCD are congruent by the *Ssa* theorem. Hence $|PD| = |BD|$, and thus $|\angle ABP| = 45°$. Denote again by E the intersection of the line BP with the side AC. In the triangle ABE the angle BEA is right, so the line BP is the altitude of the triangle ABC (Fig. 1) and the point P is therefore its orthocenter. It follows that AP is the altitude to the side BC, hence $AP \perp BC$.

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3. *Find all integers greater than 1 by which a cancellation can be made in some fraction of the form*

$$\frac{3p - q}{5p + 2q},$$

where p and q are mutually prime integers.

Solution. The fraction admits cancellation by an integer $d > 1$ if and only if d is a common divisor of its numerator and its denominator. Let us thus assume that $d \mid 3p-q$ and at the same time $d \mid 5p+2q$, where p and q are mutually prime integers. Adding up the appropriate multiples of the two binomials $3p-q$ and $5p+2q$, we get

$$2(3p-q)+(5p+2q)=11p \quad \text{and} \quad 3(5p+2q)-5(3p-q)=11q.$$

Since both $3p-q$ and $5p+2q$ are assumed to be divisible by d, the two numbers $11p$ and $11q$ must also be multiples of d. However, p and q are mutually prime by hypothesis and 11 is a prime, thus the numbers $11p$ and $11q$ have only one common divisor greater than 1, namely the number 11. Thus $d = 11$.

We now need to show that it is indeed possible to make a cancellation by 11 in some fraction of the given form. That is, we need to find a pair of mutually prime integers p and q so that $11 \mid 3p-q$ and at the same time $11 \mid 5p+2q$. Solving the system of equations

$$3p-q=11m \quad \text{and} \quad 5p+2q=11n,$$

we get $(p,q)=(2m+n, 3n-5m)$, and p, q will certainly be prime if we choose them so that $q=3n-5m=1$, thus e.g. for $n=2$ and $m=1$, when $(p,q)=(4,1)$ and the corresponding fraction is $11/22$.

Answer. The only integer greater than 1, which can be cancelled in some fraction of the given form, is the number 11.

Second Round of the 58th Czech Mathematical Olympiad (January 20th, 2009)

1. *A four-digit natural number is given which is divisible by seven. If we write its digits in the reverse order, we obtain another four-digit natural number, which is greater and is also divisible by seven. Furthermore, both numbers give the same remainder upon division by 37. Determine the original four-digit number.*

Solution. Denote the given number by $n = \overline{abcd} = 1000a + 100b + 10c + d$, and the one obtained by reversing the order of digits $k = \overline{dcba} = 1000d + 100c + 10b + a$. Since both numbers k, n give the same remainder upon dividing by 37, their difference

$$\begin{aligned} k - n &= (1000d + 100c + 10b + a) - (1000a + 100b + 10c + d) = \\ &= 999(d - a) + 90(c - b) = 37 \cdot 27(d - a) + 90(c - b) \end{aligned} \tag{1}$$

must be divisible by 37, whence $37 \mid 90(c - b)$. As 37 is a prime and 90 is not its multiple, necessarily $37 \mid c - b$. For two digits b, c, this is only possible if $b = c$. Conversely, if $b = c$, then it follows from (1) that the difference $k - n$ is divisible by 37, regardless of the values of a and d, so the two numbers k and n give the same remainder upon division by 37. From now on, we can therefore assume that $n = \overline{abbd}$ and $k = \overline{dbba}$, and will be concerned only with the conditions of the divisibility by seven.

From the conditions $7 \mid n$, $7 \mid k$ we have

$$7 \mid k - n = 37 \cdot 27(d - a)$$

(we have substituted the equality $b = c$ into (1)). Since the numbers 7 and $37 \cdot 27$ are relatively prime, we must have $7 \mid d - a$. Since we know that $k > n$, necessarily $d > a$; and as a, d are digits, this is only possible if $d - a = 7$. Finally, as a is the first digit of the four-digit number n, we must have $a > 0$; thus the only possibilities are $a = 1$, $d = 8$ or $a = 2$, $d = 9$.

For $a = 1$ and $d = 8$, the numbers

$$n = \overline{1bb8} = 1\,008 + 110b = 7 \cdot (144 + 15b) + 5b,$$
$$k = \overline{8bb1} = 8\,001 + 110b = 7 \cdot (1\,143 + 15b) + 5b,$$

are divisible by seven if and only if $7 \mid 5b$, or $b \in \{0, 7\}$. We thus get the first two solutions $n = 1\,008$ and $n = 1\,778$.

Similarly, for $a = 2$ and $d = 9$,

$$n = \overline{2\,bb9} = 2\,009 + 110b = 7 \cdot (287 + 15b) + 5b,$$
$$k = \overline{9\,bb2} = 9\,002 + 110b = 7 \cdot (1\,286 + 15b) + 5b,$$

are divisible by seven iff $7 \mid 5b$, which leads to the two remaining solutions $n = 2\,009$ and $n = 2\,779$.

The original four-digit number can be any of the numbers 1 008, 1 778, 2 009 and 2 779 (and no other).

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2. *Two circles k_a and k_b are given, whose centers lie on the legs of lengths a and b, respectively, of a right triangle. Both circles are tangent to the hypotenuse of the triangle pass through the vertex opposite to the hypotenuse. Denote the radii of the circles by ρ_a and ρ_b. Find the greatest real number p such that the inequality*

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} \geqslant p\Big(\frac{1}{a} + \frac{1}{b}\Big)$$

holds for all right triangles.

Solution. Denote the vertices of the triangle by A, B, C, with A, B lying opposite the legs with lengths a and b, respectively.

Let us first determine the magnitudes of the two radii ρ_a and ρ_b. Denote by A' the image of A in the symmetry with respect to the line BC. The circle k_a is the incircle of the triangle $A'AB$ (Fig. 1). The isosceles triangle ABA' has perimeter $o = 2(b + c)$ and area $S = ab$, so by the familiar formula the radius ϱ_a of its incircle k_a is

$$\varrho_a = \frac{2S}{o} = \frac{ab}{b+c}.$$

Similarly $\varrho_b = ab/(a + c)$.

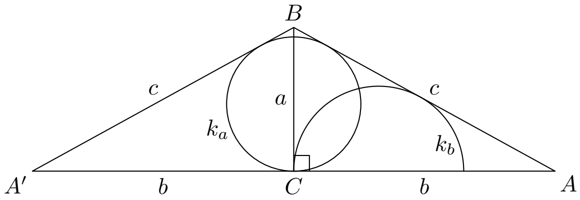

Fig. 1

The number p is to satisfy

$$p \leqslant \frac{\dfrac{1}{\varrho_a} + \dfrac{1}{\varrho_b}}{\dfrac{1}{a} + \dfrac{1}{b}} = \frac{\dfrac{b+c}{ab} + \dfrac{a+c}{ab}}{\dfrac{a+b}{ab}} = \frac{a+b+2c}{a+b} = 1 + \frac{2c}{a+b} = 1 + \frac{2\sqrt{a^2+b^2}}{a+b}$$

for any right triangle with legs a, b.

Since for $a = b$ the last expression equals $1 + \sqrt{2}$, any such number p must satisfy $p \leqslant 1 + \sqrt{2}$. If we can show that

$$\frac{2\sqrt{a^2+b^2}}{a+b} \geqslant \sqrt{2} \tag{1}$$

for any positive a and b, it will follow that $p = 1 + \sqrt{2}$ is the solution of the problem.

The inequality (1) for arbitrary positive a, b is easily shown to be equivalent to an inequality whose validity is evident:

$$\begin{aligned} 2\sqrt{a^2+b^2} &\geqslant \sqrt{2}(a+b), \\ 4(a^2+b^2) &\geqslant 2(a+b)^2, \\ 4a^2+4b^2 &\geqslant 2a^2+4ab+2b^2, \\ 2(a-b)^2 &\geqslant 0. \end{aligned}$$

Alternatively, one can appeal to Cauchy's inequality $2(a^2+b^2) \geqslant (a+b)^2$, or to the inequality between the quadratic and arithmetic mean

$$\sqrt{\frac{a^2+b^2}{2}} \geqslant \frac{a+b}{2}.$$

Answer. The sought real number p is $1 + \sqrt{2}$.

Remark. The magnitude of the radii ϱ_a, ϱ_b can be derived also in another way: expressing the sine of the angle ABC in two ways from the right triangles S_aBT a ABC (Fig. 2), we get

$$\frac{\varrho_a}{a-\varrho_a} = \frac{b}{c},$$

implying that $\varrho_a = ab/(b+c)$. Similarly for ϱ_b.

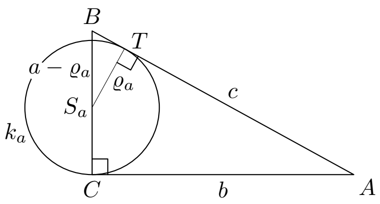

Fig. 2

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3. *Find the magnitudes of the interior angles α, β, γ of a triangle which satisfy*

$$\begin{aligned} 2\sin\beta\sin(\alpha+\beta) - \cos\alpha &= 1, \\ 2\sin\gamma\sin(\beta+\gamma) - \cos\beta &= 0. \end{aligned}$$

Solution. Similarly as in the solution of the first problem of the First Round we can derive the following formula

$$\cos x - 2\sin y \sin(x+y) = \cos(x+2y)$$

which holds for arbitrary real numbers x, y. From conditions of the given problem we get the following two equations for magnitudes of the interior angles α, β, γ of a triangle:

$$\begin{aligned}\cos\alpha - 2\sin\beta\sin(\alpha+\beta) = \cos(\alpha+2\beta) = -1,\\ \cos\beta - 2\sin\gamma\sin(\beta+\gamma) = \cos(\beta+2\gamma) = 0.\end{aligned}$$

This yields

$$\alpha + 2\beta = \pi + 2k\pi, \tag{1}$$
$$\beta + 2\gamma = \frac{\pi}{2} + l\pi, \tag{2}$$

where k, l are arbitrary non-negative integers. Adding up (1) and (2) we obtain

$$\alpha + 3\beta + 2\gamma = \frac{3\pi}{2} + (2k+l)\pi.$$

Since α, β, γ are magnitudes of interior angles of a triangle ($\alpha+\beta+\gamma=\pi$) we have

$$2\beta + \gamma = \frac{\pi}{2} + (2k+l)\pi, \tag{3}$$

where k, l are non-negative integers. Further from (2) a (3) we obtain after short manipulation

$$\beta + \gamma = \frac{\pi}{3} + \frac{2}{3}(k+l)\pi.$$

Regarding $\beta+\gamma<\pi$ we have $k=l=0$, thus

$$\beta + \gamma = \frac{\pi}{3}. \tag{4}$$

From (2) and (4) it easily follows $\beta=\gamma=\frac{1}{6}\pi$, hence $\alpha=\frac{2}{3}\pi$.

Conclusion. The magnitudes of interior angles of the considered triangle are $\alpha=120^\circ$, $\beta=\gamma=30^\circ$.

Another solution. From the equality $\alpha+\beta+\gamma=\pi$ and standard goniometric formulas we have

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\gamma,\\ \cos\alpha &= -\cos(\beta+\gamma) = -\cos\beta\cos\gamma + \sin\beta\sin\gamma.\end{aligned}$$

Substituting these values of $\sin(\alpha+\beta)$ and $\cos\alpha$ into the first equation of the given system and simplifying, we get

$$\begin{aligned}2\sin\beta\sin\gamma - (-\cos\beta\cos\gamma + \sin\beta\sin\gamma) &= 1,\\ \cos\beta\cos\gamma + \sin\beta\sin\gamma &= 1,\\ \cos(\beta-\gamma) &= 1.\end{aligned}$$

The last equality holds if and only if $\beta = \gamma$, since the difference of two interior angles of a triangle always lies in the interval $(-\pi, \pi)$, in which the cosine functions assume the value 1 only at the point zero. We have thus shown that the first equation of the given system is fulfilled if and only if $\beta = \gamma$.

Now it is easy to solve also the second equation of the system, if we make the substitution $\beta = \gamma$ in it:

$$\begin{aligned} 2\sin\beta\sin 2\beta - \cos\beta &= 0, \\ 4\sin^2\beta\cos\beta - \cos\beta &= 0, \\ (4\sin^2\beta - 1)\cos\beta &= 0. \end{aligned}$$

Hence either $\cos\beta = 0$, or $\sin\beta = \pm\frac{1}{2}$. However, the equality $\beta = \gamma$ for angles in a triangle means that the angle β is acute, hence $\cos\beta > 0$. Thus $\sin\beta = \frac{1}{2}$ (the possibility $\sin\beta = -\frac{1}{2}$ is ruled out for an angle β in the interval $(0, \pi)$). We have thus arrived at the only possible values $\beta = \gamma = 30°$, implying $\alpha = 120°$. These values are indeed a solution of the original system, since the first equation holds owing to the equality $\beta = \gamma$, while the second has been treated under the assumption of $\beta = \gamma$ only by equivalent manipulations.

---

4. *A point D has been chosen in the interior of the side BC of an acute triangle ABC, and another point P in the interior of the segment AD but not lying on the median from the vertex C. The line containing this median intersects the circumcircle of the triangle CPD at a point which we denote by K ($K \neq C$). Show that the circumcircle of the triangle AKP passes in addition to A through another fixed point which does not depend on the choice of the points D and P.*

Solution. Denote by ϕ the magnitude of the angle between the line t_c containing the median from the vertex C and the line containing the side BC of the given triangle. In view of the definition of the point K, the lines KP and AD will meet also at the same angle ϕ. This means, however, that on the circumcircle of the triangle AKP there also lies the point M of the line t_c at which the line AM meets the line t_c at the angle ϕ. This property is clearly possessed by the point M symmetric to C with respect to the midpoint of the side AB; and this point does not depend on the choice of the points D and P (Fig. 3).

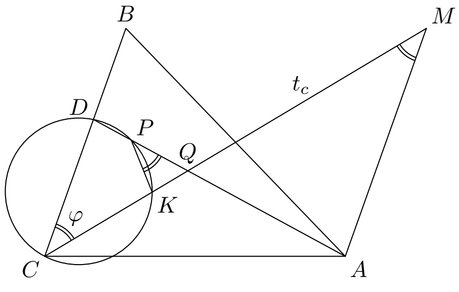

Fig. 3

Let us now show the above in more detail. Denote by Q the intersection of the median t_c with the segment AD (that is, Q is the "forbidden" choice for the point P). The point P lies either in the interior of the segment DQ (Fig. 3), or in the interior of the segment QA (Fig. 5).

In the former case, the point Q lies in the exterior of the circumcircle of the triangle CPD, and the point K thus belongs to the interior of the half-line QC. If the point K lies in the interior of the segment QC, the points C and P are the opposite vertices of the cyclic quadrangle $CDPK$, thus $|\angle APK| = \phi$. Furthermore, the points P and M lie in the same half-plane determined by the line AK, and from the congruence of the angles AMK and APK it therefore follows that the quadrangle $AMPK$ is cyclic, so the point M indeed lies on the circumcircle of the triangle AKP.

If the point K does not lie in the interior of the segment QC and $K \neq C$ (Fig. 4), then $|\angle KPD| = |\angle KCD| = 180° - \phi$, whence $|\angle KPA| = \phi = |\angle KMA|$. (The last equality holds, of course, also for $K = C$.) Since the points P and M lie in the same half-plane determined by the line KA, the point M lies also in this case on the circumcircle of the triangle AKP.

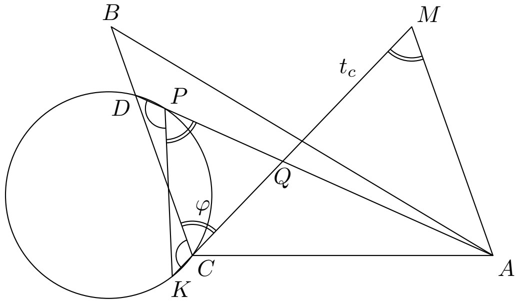

Fig. 4

In the second case the point K lies in the interior of the half-line QM. If K lies in the interior of the segment QM (Fig. 5), the points P and M lie in opposite half-planes determined by AK, and from the equality of the angles DCK and DPK

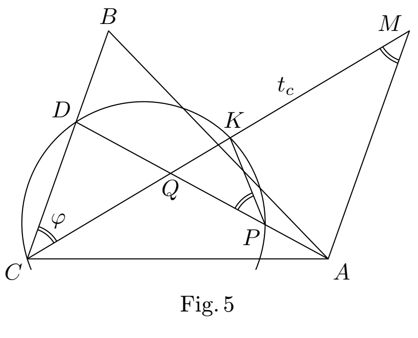

Fig. 5

subtending the chord DK we have $|\angle DPK| = \phi = |\angle AMK|$, which guarantees that the quadrangle $AMKP$ is cyclic, so that the point M lies on the circumcircle of the triangle AKP.

If the point K does not lie in the interior of the segment QM (Fig. 6), we have $|\angle KPA| = |\angle KMA| = 180° - \phi$. Since the points P and M lie in the same half-plane determined by KA, the point M lies also in this case on the circumcircle of the triangle AKP.

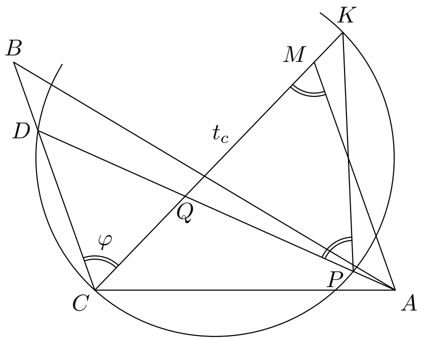

Fig. 6

Final Round of the 58th Czech Mathematical Olympiad (March 23–24, 2009)

1. *Show that if the numbers p, $3p+2$, $5p+4$, $7p+6$, $9p+8$ and $11p+10$ are all primes, then the number $6p+11$ is composite.*

Solution. Assume that all of the numbers p, $3p+2$, $5p+4$, $7p+6$, $9p+8$ a $11p+10$ are primes. Let us see what are the possible remainders of p upon division by five, i.e. what numbers l from the set $\{0,1,2,3,4\}$ and nonnegative integers k can satisfy $p=5k+l$.

- ▷ If $p=5k$ is a prime, then $p=5$, but then $11p+10=65$ is not a prime.
- ▷ If $p=5k+1$, then $3p+2=5(3k+1)$ is a prime only if $k=0$, but then $p=1$ which is not a prime.
- ▷ If $p=5k+2$, then $7p+6=5(7k+4)$ cannot be a prime for any $k \geqslant 0$.
- ▷ If $p=5k+3$, then $9p+8=5(9k+7)$ cannot be a prime for any $k \geqslant 0$.

The prime p thus must be of the form $5k+4$. But then $6p+11=5(6k+7)$ is a composite number, for any integer $k \geqslant 0$.

Remark. The least prime p for which all the numbers $3p+2$, $5p+4$, $7p+6$, $9p+8$ and $11p+10$ are also primes is $p=2\,099$.

2. *On the shorter of the arcs CD of the circumcircle of a rectangle $ABCD$, a point P is chosen. Denote the feet of the perpendiculars from P onto the lines AB, AC and BD by K, L and M, respectively. Show that the angle LKM has magnitude $45°$ if and only if $ABCD$ is a square.*

Solution. We will show that the angle LKM has the same magnitude as the angle CBD (Fig. 1). This already implies the desired assertion in a trivial way (the angle CBD has magnitude $45°$ if and only if $|BC|=|CD|$, i.e. if and only if $ABCD$ is a square).

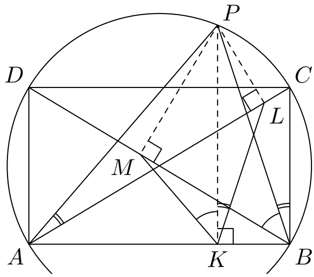

Fig. 1

The points B, K, M, P lie, in this order, on the Thaletian circle over the diameter BP. For the magnitudes of the angles subtending the chord PM we thus have $|\angle PKM| = |\angle PBM|$. Similarly the points A, K, L, P lie (in this order) on the Thaletian circle over the diameter AP and for the angles subtending the chord PL we have $|\angle LKP| = |\angle LAP|$. Finally, for the magnitudes of the angles subtending the chord CP in the circumcircle of the rectangle $ABCD$ we obtain $|\angle CAP| = |\angle CBP|$.

Combining these equalities yields

$$|\angle LKM| = |\angle LKP| + |\angle PKM| = |\angle LAP| + |\angle PBM| = |\angle CAP| + |\angle PBD| = \\ = |\angle CBP| + |\angle PBD| = |\angle CBD|,$$

which is what we wanted to prove.

Remark. The above argument applies also in the trivial case when $P = C$ or $P = D$; some of the angles considered will then have zero magnitude.

Another solution. We again show that the angles LKM and CBD have the same magnitude. Denote by N the foot of the perpendicular from the point P onto the line BC. The points K, L, N lie on Simpson's line corresponding to the point P and triangle ABC (Fig. 2). The Thaletian circle over diameter PB contains the points K, M and N. By the magnitudes of the angles subtending the chord MN of the same circle we thus get

$$|\angle LKM| = |\angle NKM| = |\angle NBM| = |\angle CBD|.$$

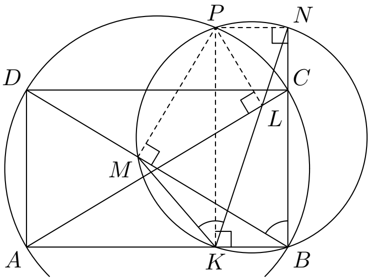

Fig. 2

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3. *Find the least positive number x with the following property: if a, b, c, d are arbitrary positive numbers whose product is 1, then*

$$a^x + b^x + c^x + d^x \geqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

Solution. Let a, b, c, d be positive numbers whose product equals 1. From the inequality between the arithmetic and geometric mean of the three numbers a^x, b^x, c^x, with arbitrary $x > 0$, we obtain

$$\frac{a^x + b^x + c^x}{3} \geqslant \sqrt[3]{a^x b^x c^x} = \sqrt[3]{\frac{1}{d^x}}.$$

Choosing $x = 3$ gives the inequality $\frac{1}{3}(a^3 + b^3 + c^3) \geqslant 1/d$. Similarly,

$$\tfrac{1}{3}(a^3 + b^3 + d^3) \geqslant 1/c, \quad \tfrac{1}{3}(a^3 + c^3 + d^3) \geqslant 1/b, \quad \tfrac{1}{3}(b^3 + c^3 + d^3) \geqslant 1/a.$$

Adding up these four inequalities we obtain

$$a^3 + b^3 + c^3 + d^3 \geqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d},$$

so for $x = 3$ the given inequality always holds.

We now show that $x = 3$ is the smallest number with this property, that is, that for any positive $x < 3$ there is some quadruple (a, b, c, d), $abcd = 1$, for which the given inequality fails. We will look for such a quadruple in the form $a = b = c = t$ and $d = 1/t^3$ with suitable $t > 1$ (depending on $x < 3$). Positive numbers a, b, c, d like this certainly satisfy $abcd = 1$, while

$$a^x + b^x + c^x + d^x = 3t^x + \frac{1}{t^{3x}} < 4t^x \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{3}{t} + t^3 > t^3.$$

Thus if we can choose t so that $4t^x < t^3$, then the desired inequality will be violated. In view of the condition $x < 3$, the inequality $4t^x < t^3$ is equivalent to

$$t > 4^{\frac{1}{3-x}},$$

which is certainly fulfilled for t large enough.

Conclusion. The least number with the desired property is $x = 3$.

---

4. *We are interested in natural numbers (positive integers) n with the property that there exist exactly four natural numbers k such that $n + k$ is a divisor of $n + k^2$.*

a) Show that $n = 58$ has this property, and find the corresponding four numbers k.

b) Show that an even number $n = 2p$, where $p \geqslant 3$, has this property if and only if both p and $2p + 1$ are primes.

Solution. From the equality $n+k^2 = (k+n)(k-n)+n(n+1)$ we see that $n+k \mid n+k^2$ is equivalent to $n + k \mid n(n + 1)$. The number of such k is equal to the number of divisors of $D = n(n + 1)$ which are greater than n.

a) For $n = 58$, the prime factorization of the corresponding $D = 58 \cdot 59 = 2 \cdot 29 \cdot 59$ reveals that the divisors of D greater than 58 are precisely the four numbers 59, $2 \cdot 59 = 118$, $29 \cdot 59 = 1\,711$ and $2 \cdot 29 \cdot 59 = 3\,422$. These are equal to $58 + k$, so the corresponding four values of k are, in turn, $k = 1$, $k = 60$, $k = 1\,653$ and $k = 3\,364 = 58^2$. (We remark that the two numbers $k = 1$ and $k = n^2$ satisfy the condition $n + k \mid n + k^2$ for any n.)

b) For an even $n = 2p$, where $p \geqslant 3$, we have $D = 2p(2p + 1)$, and we can easily list four divisors of D which are greater than $n = 2p$:

$$2p + 1 < 2(2p + 1) < p(2p + 1) < 2p(2p + 1). \qquad (1)$$

If both p and $2p+1$ are primes, that it is clear that D cannot have any other divisors larger than n, and thus $n = 2p$ has the desired property.

On the other hand, if at least one of the numbers p, $2p+1$ is composite and $p \geqslant 3$, we will show that the corresponding number D has at least one more divisor greater than n in addition to those listed in (1). We distinguish two cases, depending on which of the numbers p, $2p+1$ is composite.

(i) If p is composite, then it is divisible by some q, $2 \leqslant q \leqslant \frac{1}{2}p$, and the number D has $2q(2p+1)$ as a divisor. If $q \neq \frac{1}{2}p$, then this divisor lies between the second and the third of the divisors listed in (1):

$$2(2p+1) < 2q(2p+1) < p(2p+1).$$

If $q = \frac{1}{2}p$ is the only nontrivial divisor of p, then necessarily $p = 4$, thus $2p+1 = 9$ is also composite, and we can continue as in the part (ii) below.

(ii) If the (odd) number $2p+1$ is composite, then it is divisible by some q, $3 \leqslant q < p$, and the number D has $2pq$ as a divisor which lies between the second and the third of the divisors listed in (1):

$$2(2p+1) < 2pq < p(2p+1), \quad \text{since} \quad q > 2 + \frac{1}{p} \ \text{ and } \ q < p + \frac{1}{2}.$$

This completes the proof of the part b).

---

5. *In each vertex of a regular n-gon $A_1A_2\dots A_n$ there lies a certain number of coins: in the vertex A_k there are exactly k coins, for each $1 \leqslant k \leqslant n$. We choose two coins and move each of them into one of the neighbouring vertices in such a way that one is moved clockwise and the other anti-clockwise. Decide for which n it is possible to achieve, after a finite number of steps, that for each k, $1 \leqslant k \leqslant n$, there are exactly $n+1-k$ coins in the vertex A_k.*

Solution. We assign to each coin the index i of the vertex A_i in which it lies (thus $i \in \{1, 2, \dots, n\}$), and we update these numbers after each movement. Let us observe how the sum S of all these numbers can change upon a single movement.

If no coin is moved between the pair of vertices A_1 and A_n, then the sum S remains unchanged, since one of the numbers assigned to the coins increases by one, while the second one decreases by one (and the other are unaffected). Similarly, S does not change if we move one coin from A_1 into A_n, while moving the other coin from A_n into A_1. If one coin is moved from A_1 into A_n and the other from A_i into A_{i+1} (where $1 \leqslant i \leqslant n-1$), then the sum S is increased by $(n-1)+1 = n$. Finally, if we move one coin from A_n into A_1 and the other coin from A_i into A_{i-1} (where $2 \leqslant i \leqslant n$), then the sum S decreases by n. In summary, it follows that *the remainder of S upon division by n does not change.*

In the original position, the sum S equals

$$1 \cdot 1 + 2 \cdot 2 + \cdots + n \cdot n = \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1),$$

while in the desired final position, it would be equal to

$$\sum_{k=1}^{n} k(n+1-k) = (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 =$$
$$= \frac{1}{2}n(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}n(n+1)(n+2)$$

(we have used the familiar formulas for the sums of the first n natural numbers and of their squares). Thus these two values of S should give the same remainder upon division by n, i.e. their difference $\frac{1}{6}n(n+1)(n-1)$ should be divisible by n. The number $\frac{1}{6}(n+1)(n-1) = \frac{1}{6}(n^2-1)$ thus must be an integer. Checking all the possible values 0 to 5 of the remainders mod 6 we find that this happens if and only if n has remainder either 1 or 5 upon division by 6. In the rest of this solution, we show that for all such n it is indeed possible to move the coins into the desired position.

Let us denote the (single) coin which is originally in the vertex A_1 by M. We will be moving all coins except M in the same common direction, while the only coin moved in the opposite direction will be M (and we will not be bothered about the actual position of M during the process). The final position of M will be determined by the above property of the sum S: the index i of the vertex A_i, in which M finds the final position, is uniquely determined by the fact that the final value of the sum S must give the same remainder upon division by n as its initial value—this is so because the indices of the vertices form a complete system of remainders mod n.

Using the above procedure of permanently moving the coin M, we can achieve that any one of the other coins can be moved into any vertex we choose (without changing the locations of the remaining coins, except M). After a finite number of steps, we can thus achieve that there are exactly $n+1-i$ coins different from M in the vertex A_i, for each $i > 1$; in the remaining vertex A_1, there will be $n-1$ coins different from M (since the total number of coins remains the same all the time), while the coin M ends up in some—as yet unknown—vertex. If we can show that this vertex is A_1, then we are finished. However, this happens if and only if the number n of vertices is such that the sum S gives the same remainder upon division by n in the initial and in the final position. And we have found all such n in the fist part of the solution.

Answer. The coins can be moved into such position if and only if n gives remainder either 1 or 5 upon division by six.

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6. *Two distinct points O and T are given in the plane ω. Find the locus of vertices of all triangles lying in ω whose centroid is T and whose circumcenter is O.*

Solution. Consider a point A in the plane ω. In order that A be a vertex of a triangle as required, A must be different from T and O. We first describe how to construct a triangle ABC if its vertex A, its centroid T and its circumcenter O are given (for three mutually distinct points A, O, T). Then we check for which points A it is impossible to construct such triangle.

Denote by A' the midpoint of the side BC. The point A' is the image of A in the homothety with center T and coefficient $-\frac{1}{2}$. If $A' \neq O$, the points B and C lie on the perpendicular p through A' to the line OA', and at the same time on the circumcircle k with center O and radius $|OA|$ (Fig. 3).

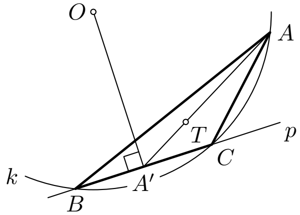

Fig. 3

For a given A, it is always possible to construct its image A' in the homothety as above. Assume first that $A' \neq O$. In order to obtain two different points B and C, the line p must be a secant of the circle k. This happens if and only if $|OA'| < |OA|$. Denote by O' the image of O in the homothety with center T and coefficient -2. Then $|O'A| = 2|OA'|$, so the above condition can be expressed as $|O'A| < 2|OA|$. The point A must therefore lie in the exterior of the Apollonian circle $m(S; |ST|)$, where S is the point symmetric to T with respect to O (Fig. 4).

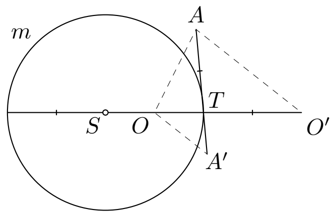

Fig. 4

Thus if $A' \neq O$, or $A \neq O'$, the construction yields three points A, B, C. These will be vertices of a triangle as required, provided they are not collinear. They are collinear if and only if the line BC coincides with the line AT, i.e. if and only if the line OA' is perpendicular to AT. The point A' thus must not lie on the Thaletian circle over diameter OT, i.e. (upon applying the homothety with center T and coefficient -2) the point A must not lie on the Thaletian circle with diameter $O'T$ (Fig. 5).

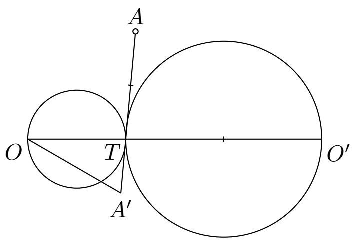

Fig. 5

In the case when the point A coincides with O', i.e. $A' = O$, we can take instead of the perpendicular p any line (different from AT) passing through O (Fig. 6). In this way we obtain infinitely many different triangles ABC with right angles at the vertex A which satisfy the conditions of the problem.

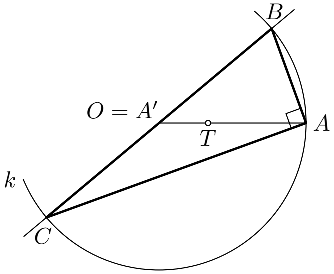

Fig. 6

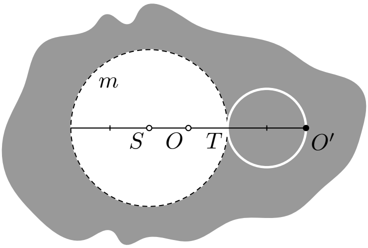

Fig. 7

Conclusion. The sought locus is the exterior of the circle m, except for the points lying on the Thaletian circle with diameter $O'T$, but including the point O' (Fig. 7).

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

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