2012

61st Czech and Slovak Mathematical Olympiad

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First Round of the 61st Czech and Slovak Mathematical Olympiad
Problems for the take-home part
(October 2011)

1. *Let n be the sum of all ten-digit numbers, which contain every decimal digit from zero to nine. Determine the remainder of n when divided by 77.*

(Pavel Novotný)

Solution. We determine the value of n first. We count for each non-zero digit the number of its occurrences in the summands (giving n altogether) in the place of units, tens, hundreds, … This enables us to evaluate the “contribution” of each digit to the sum, and thus n itself. Every digit is on each of the places $9! - 8!$ times (we have to subtract the “numbers” starting with zero) except for the first place, where it is $9!$ times (we do not have to subtract anything). In total we get

$$\begin{aligned} n &= (1+2+\cdots+9)(9!-8!)(10^8+10^7+\cdots+10+1) + (1+2+\cdots+9)\cdot 9!\cdot 10^9 \\ &= 45(8\cdot 8!\cdot 111\,111\,111 + 9\cdot 8!\cdot 10^9) = 45\cdot 8!\cdot 9\,888\,888\,888. \end{aligned}$$

Thus n is divisible by 7 and by the criterion of divisibility by 11 is

$$9\,888\,888\,888 \equiv 8-8+8-8+\cdots+8-9 \equiv -1 \pmod{11}.$$

Further $8! \equiv 5 \pmod{11}$. In total

$$n \equiv 45\cdot 8!\cdot 9\,888\,888\,888 \equiv 1\cdot 5\cdot 10 \equiv 6 \pmod{11}.$$

Finally $n \equiv 28 \pmod{77}$.

2. *There are several people on a party. Every two persons, which do not know each other, have exactly two common friends. A and B know each other but do not have any common friend. Show that A and B have the same number of friends at the party. Show that it could be exactly six persons on the party. (As usual the relation of friendship is symmetric, and “know someone” = “be friend with”.)*

(Vojtech Bálint)

Solution. Let M_A be the set of friends of A, M_B the set of friends of B (Fig. 1). We

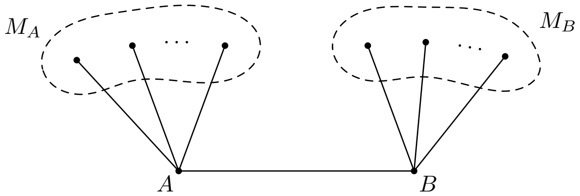
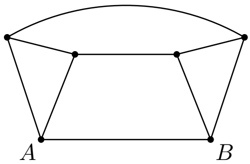


Fig. 1

show $|M_A| \geqslant |M_B|$. No person from M_A knows B (otherwise A and B would have common friend). That is everyone from M_A, let's say X, has two common friends with B. One of them is A and the other one has to be from M_B (it is the set of all B's friends), let us call him (her) X_B. Then for any X, $Y \in M_A$ we have $X_B \neq Y_B$ (otherwise X, Y, and B would be three common friends of X_B and A; again, no person from M_B knows A). Thus $M_A \geqslant M_B$. From symmetry $M_B = M_A$.

The appropriate situation for the six persons is on the following diagram:



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3. *Let S be the incenter, T the centroid, and V the orthocenter of a triangle.*

a) Show, that S is an interior point of the segment TV.

b) Determine the ratio of the side lengths of the triangle, if S is the midpoint of TV. (Jaromír Šimša)

Solution. a) Let A, B, and C be the vertices of the triangle with the basis BC Let H be the midpoint of BC, and further let $a = BC$, $b = AC = AB$, $\beta = \gamma = 90^\circ - \frac{1}{2}\alpha$. Further let P, D, and M be intersections of the height, the bisector, and the median from B with AC, respectively.

If $b > a$ (see Fig. 2a), then $\beta > 60^\circ$ and

$$\angle CBP = 90^\circ - \beta < 90^\circ - 60^\circ = 30^\circ < \tfrac{1}{2}\beta = \angle CBD,$$

thus $CP < CD$. As well known, the bisector from B divides the opposite side in the ratio a/b, that is $CD/AD = a/b < 1$, and $CD < \frac{1}{2}CA = CM$. Summing up, $CP < CD < CM$ and D is inside MP.

If $a > b$ (see Fig. 2b), then $\beta < 60^\circ$ and similarly as in the previous case $\angle CBP = 90^\circ - \beta > 90^\circ - 60^\circ = 30^\circ > \frac{1}{2}\beta = \angle CBD$, $CD/AD = a/b > 1$, thus $CP > CD > \frac{1}{2}CA = CM$, and D is inside MP again.

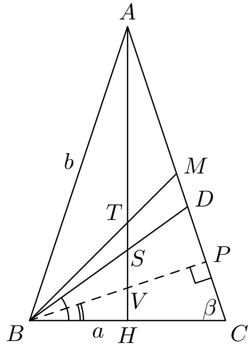

Fig. 2a

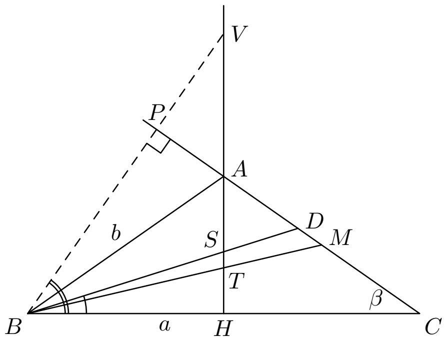

Fig. 2b

b) First we express TH, SH, VH using a, b, c, and v (the altitude from A of the triangle ABC from A).

From the properties of the centroid we have $TH = \frac{1}{3}v$.

SH is the inradius, thus $S_{ABC} = \varrho \cdot s$ with $s = \frac{1}{2}(a + 2b)$.

$$SH = \varrho = \frac{S_{ABC}}{s} = \frac{\frac{1}{2}av}{\frac{1}{2}(a+2b)} = \frac{av}{a+2b}.$$

The triangles BVH and ABH are similar (both are right-angled with $\angle VBH = 90° - \beta = \frac{1}{2}\alpha = \angle BAH$). Thus $VH/BH = BH/AH$ and

$$VH = \frac{BH^2}{AH} = \frac{a^2}{4v}.$$

Now S, T, V lie on the ray HA therefore $TS = SV$ is equivalent to

$$TH + VH = 2SH,$$

After easy manipulation we get (using $v^2 = b^2 - \frac{1}{4}a^2$):

$$(2a - b)(a - b) = 0.$$

There is $a \neq b$ (according the statement of the problem), thus S is the midpoint of the segment TV if and only if $2a = b$, that is the ratio of the side lengths is $1 : 2 : 2$.

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4. *Let p, q be two different primes, let m and n be positive integers such that the sum*

$$\frac{mp-1}{q} + \frac{nq-1}{p}$$

is integer. Prove

$$\frac{m}{q} + \frac{n}{p} > 1.$$

(Jaromír Šimša)

Solution. We rewrite:

$$\frac{mp-1}{q} + \frac{nq-1}{p} = \frac{p(mp-1) + q(nq-1)}{pq}.$$

The last fraction is a positive integer, thus $p \mid nq - 1$ and $q \mid mp - 1$, that is $pq \mid mp + nq - 1$. Therefore $mp + nq > pq$ which is the inequality in question.

5. *There are two circles, both with the radius equal to the distance of their centers. Let A and B be intersections of these circles. We choose C on k_2 such, that the segment BC meets again k_1 in L. Line AC meets k_1 again in K. Prove, that the line of the median through C of the triangle KLC passes through a fixed point, which is independent of the position of C.* (Tomáš Jurík)

Solution. Let S_1, S_2 be the centers of k_1, k_2 respectively. Let P be on k_1 such that PS_2 is a diameter of k_1. We show, that P is the sought point, namely we prove that the midpoint of KL is collinear with P and C.

Let Q be the point reflection of S_1 with respect to S_2. That is S_1Q is a diameter of k_2 and the angle S_1BQ is right, thus BQ is tangent to k_1. According to the statement of the problem, C has to be inside the shorter arc AQ of k_2. If we consider the reflection along S_1S_2 we see that PA is tangent to k_1 as well, that is K is inside the shorter arc PA of k_1 (Fig. 3).

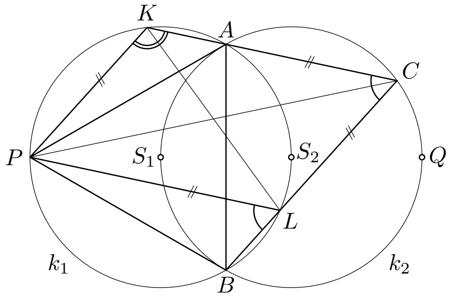

Fig. 3

Since k_1 and k_2 have the same radii, the triangles S_1S_2A and S_1S_2B are equilateral and $\angle BS_1A = 120°$. The corresponding inscribed angle BPA is therefore $60°$. Moreover A and B are symmetric along PS_2, thus $PA = PB$ and APB is equilateral. Any angle that subtends AP, PB, or BA in k_1 (or AB in k_2 as well) is therefore $60°$. Thus

$$\angle ACB = 60°, \qquad \angle PLB = 60°, \qquad \angle PKA = 120°.$$

The equality of the first two angles implies PL and KC are parallel, the sum of the first and third angle is $180°$ therefore PK and LC are parallel as well. Thus $PLCK$ is parallelogram and since diagonals in any parallelogram half each other we are done.

6. *Find the greatest real k such that*

$$\frac{2(a^2 + kab + b^2)}{(k+2)(a+b)} \geqslant \sqrt{ab}$$

holds for any positive real a and b. (Ján Mazák)

Solution. If $k = 2$ then the inequality is equivalent to $\frac{1}{2}(a+b) \geqslant \sqrt{ab}$ (which holds), therefore $k \geqslant 2$. For $k > 2$ we have $k + 2 > 0$

$$2(a^2 + kab + b^2) \geqslant (k+2)(a+b)\sqrt{ab},$$

the division by b^2 gives

$$2\Big(\frac{a^2}{b^2} + k\,\frac{a}{b} + 1\Big) \geqslant (k+2)\left(\frac{a}{b} + 1\right)\sqrt{\frac{a}{b}}.$$

Denote $\sqrt{a/b} = x$. The value of x can be any positive number. Thus the problem is equivalent to find greatest k, such that

$$2(x^4 + kx^2 + 1) \geqslant (k+2)(x^2+1)x$$

holds for any positive x. We get

$$\begin{aligned} k\big((x^2+1)x - 2x^2\big) &\leqslant 2\big(x^4 + 1 - (x^2+1)x\big), \\ k(x^3 - 2x^2 + x) &\leqslant 2(x^4 - x^3 - x + 1), \\ kx(x^2 - 2x + 1) &\leqslant 2\big(x^3(x-1) - (x-1)\big), \\ kx(x-1)^2 &\leqslant 2(x-1)^2(x^2+x+1). \end{aligned}$$

For $x = 1$ the inequality holds. If $x \neq 1$ then the division by positive number $x(x-1)^2$ gives

$$k \leqslant \frac{2(x^2+x+1)}{x} = 2 + 2\Big(x + \frac{1}{x}\Big). \tag{1}$$

If $x \neq 1$ then the values of $x + 1/x$ fill the whole interval $(2, \infty)$, the RHS can be any number in $(6, \infty)$, consequently $k \geqslant 6$.

First Round of the 61st Czech and Slovak Mathematical Olympiad (December 6th, 2011)

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1. *In the domain of real numbers solve the following system of equations*

$$y + 3x = 4x^3,$$
$$x + 3y = 4y^3.$$

(Pavel Calábek)

Solution. The sum and the difference of the equations gives

$$4(x + y) = 4(x^3 + y^3),$$
$$2(x - y) = 4(x^3 - y^3),$$

or

$$\begin{aligned} x + y &= (x + y)(x^2 - xy + y^2), \\ \tfrac{1}{2}(x - y) &= (x - y)(x^2 + xy + y^2). \end{aligned} \tag{1}$$

If $x + y = 0$, then the substitution $y = -x$ into the original first equation gives $2x = 4x^3$, or $x(2x^2 - 1) = 0$. This equation has three roots $x = 0$ and $x = \pm\frac{1}{2}\sqrt{2}$, thus we get three solutions $(0, 0)$, $\left(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right)$, and $\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$.

If $x - y = 0$, then substituting $y = x$ we get $4x = 4x^3$, or $x(x^2 - 1) = 0$ and we get two new solutions $(1, 1)$ and $(-1, -1)$.

Now let $x + y$ and $x - y$ be both non-zero. Then

$$\begin{aligned} 1 &= x^2 - xy + y^2, \\ \tfrac{1}{2} &= x^2 + xy + y^2. \end{aligned} \tag{2}$$

The sum and the difference of the equations is $x^2 + y^2 = \frac{3}{4}$ and $xy = -\frac{1}{4}$, thus

$$(x + y)^2 = x^2 + y^2 + 2xy = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4},$$

and $x + y = \frac{1}{2}$ or $x + y = -\frac{1}{2}$, it means x, y are then the solutions of the quadratic equations

$$t^2 - \tfrac{1}{2}t - \tfrac{1}{4} = 0, \qquad \text{resp.} \qquad t^2 + \tfrac{1}{2}t - \tfrac{1}{4} = 0$$

with roots $t_{1,2} = \frac{1}{4} \pm \frac{1}{4}\sqrt{5}$, resp. $t_{3,4} = -\frac{1}{4} \pm \frac{1}{4}\sqrt{5}$, and we get four other solutions: (t_1, t_2), (t_2, t_1), (t_3, t_4), (t_4, t_3).

Conclusion. The solutions of the problem are

$$\begin{gathered} (0, 0),\ \left(\tfrac{1}{2}\sqrt{2}, -\tfrac{1}{2}\sqrt{2}\right),\ \left(-\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2}\right), (1, 1),\ (-1, -1), \\ \left(\tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}, \tfrac{1}{4} - \tfrac{1}{4}\sqrt{5}\right),\ \left(\tfrac{1}{4} - \tfrac{1}{4}\sqrt{5}, \tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}\right), \\ \left(-\tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}, -\tfrac{1}{4} - \tfrac{1}{4}\sqrt{5}\right),\ \left(-\tfrac{1}{4} - \tfrac{1}{4}\sqrt{5}, -\tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}\right). \end{gathered} \tag{3}$$

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2. *Let AB and CD be the basis and M the midpoint of the diagonal AC of a trapezoid $ABCD$. Prove that if ABM and ACD have the same area then DM and BC are parallel.* (Jaroslav Švrček)

Solution. BM is the median of ABC (Fig. 1), thus BM cuts ABC into two triangles with the same area. One of them (ABM) has the same area as ACD, that is the area of ABC is the double of the area of ACD. These triangles have the same altitudes on AB and on CD respectively (the height of the trapezoid), therefore $AB = 2CD$.

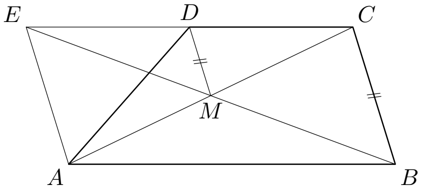

Fig. 1

Let E be on the line CD such, that D is the midpoint of CE. Since $AB = 2CD$ we have $CE = AB$, that is $ABCE$ is a parallelogram. We know M is the midpoint of BE therefore it is the midpoint of BE as well. Thus DM is a mid-segment of BCE, therefore DM is parallel to BC.

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3. *Find all positive integers n such that $(2^n + 1)(3^n + 2)$ is divisible by 5^n.* (Ján Mazák)

Solution. The following table shows 2^n+1 and 3^n+2 modulo 5 (we know, that both sequences modulo 5 have to be periodic):

n	1	2	3	4	5	6	7	8	...
2^n+1	3	5	9	17	33	65	129	257	...
modulo 5	3	0	4	2	3	0	4	2	...
3^n+2	5	11	29	83	245	731	2 189	6 563	...
modulo 5	0	1	4	3	0	1	4	3	...

Both sequences modulo 5 are periodic with period 4. But $2^n + 1$ is divisible by 5 iff n is 2 modulo 4, and $3^n + 2$ is divisible by 5 iff n is 1 modulo 4. Thus if 5^n should divide $(2^n + 1)(3^n + 2)$, it has to divide one of the factors. But for $n \geqslant 2$ obviously $5^n > 3^n + 2$ and $5^n > 2^n + 1$ and 5^n cannot divide any of the factors. For $n = 1$ we have $5^1 \mid (2^1 + 1)(3^1 + 2) = 15$, thus $n = 1$ is the unique solution of the problem.

Second Round of the 61st Czech and Slovak Mathematical Olympiad (January 17th, 2012)

1. *Let S_n denotes the sum of all n-digit numbers, which contain only digits 1, 2, 3, at least once each. Find all integers $n \geqslant 3$ such that S_n is divisible by 7.* (Pavel Novotný)

Solution. Similarly as in Problem 1 of the take-home part, we compute the sum by summing up all the contributions by all digits. The number of occurrences of any of the digits on an arbitrary but given place in the number is $k = 3^{n-1} - 2 \cdot 2^{n-1} + 1$ (we fix one position, if all the others were arbitrary, we would get 3^n occurrences, but we have to subtract numbers consisting only of two or one digit). Then the contribution p of the digit 1 to the sum is

$$p = k + 10k + 100k + \cdots + 10^{n-1}k = (1 + 10 + 100 + \cdots + 10^{n-1})k = \frac{10^n - 1}{9}\,k.$$

The contributions of the digit 2 and 3 are obviously $2p$ and $3p$, thus

$$S_n = p + 2p + 3p = 6p = 6 \cdot \frac{10^n - 1}{9}\,k = \frac{2}{3}(10^n - 1)(3^{n-1} - 2^n + 1).$$

S_n is divisible by 7 iff at least one of the factors $10^n - 1$, $3^{n-1} - 2^n + 1$ is. The following table lists the factors modulo 7 for a few small integers:

n	1	2	3	4	5	6	7	8
$10^n - 1$	9	99	999	9 999	99 999	999 999	...	
modulo 7	2	1	5	3	4	0	2	1
$3^{n-1} - 2^n + 1$	0	0	2	12	50	180	602	1 932
modulo 7	0	0	2	5	1	5	0	0

Both sequences modulo 7 are periodic with the period 6 (10^n modulo 7 is periodic with the period 6, so is $10^n - 1$; 3^n modulo 7 is periodic with the period 6 and 2^n modulo 7 has period 3, thus $3^{n-1} - 2^n + 1$ has period 6) and $10^n - 1$ is divisible by 7 for $n = 6k$, $3^{n-1} - 2^n + 1$ is a multiple of 7 for $n = 6k + 1$ or $n = 6k + 2$.

Conclusion. All solutions of the problem are integers of the form $6m$, $6m + 1$, and $6m + 2$ for any positive integer m.

2. *Find an arithmetic progression with the first element an integer $a > 1$ such that exactly two numbers from a^2, a^3, a^4, a^5 are elements of the progression as well and it's difference is as large as possible (the difference does not have to be an integer).* (Jaromír Šimša)

Solution. An arithmetic progression $\mathcal{A}$ with the first element a and a positive difference d contains those numbers from a^2, a^3, a^4, a^5 for which the corresponding difference from

$$\begin{aligned} a^2 - a &= a(a-1), \\ a^3 - a &= a(a-1)(a+1), \\ a^4 - a &= a(a-1)(a^2+a+1), \\ a^5 - a &= a(a-1)(a+1)(a^2+1) \end{aligned} \tag{1}$$

is an integer multiple of d.

Let us suppose $\mathcal{A}$ contains just two numbers out of a^2, a^3, a^4, a^5. If $a^2 \in \mathcal{A}$, then $a(a-1)$ is an integer multiple of d. Since a is an integer, then all the differences in (1) (all of them are integer multiples of $a(a-1)$) are integer multiples of d. Thus $\mathcal{A}$ contains all numbers a^2, a^3, a^4, a^5, therefore the sought $\mathcal{A}$ does not contain a^2, and it contains exactly two from a^3, a^4, a^5.

If $a^3 \notin \mathcal{A}$, or $a^4, a^5 \in \mathcal{A}$, then

$$\frac{a^4 - a}{d}, \qquad \frac{a^5 - a}{d}$$

are integers.

Then also

$$\frac{a^5 - a}{d} - a \cdot \frac{a^4 - a}{d} = \frac{a^2 - a}{d},$$

is integer. Consequently $a^2 \in \mathcal{A}$, contradiction. Thus $a^3 \in \mathcal{A}$. Then we have $d \leqslant a^3 - a$ for the difference d, where the equation holds (d is the largest one) if a^3 is the second element of the progression. Then $a^2 \notin \mathcal{A}$, and $a + (a^2+1)(a^3-a) = a^5 \in \mathcal{A}$. Also

$$\frac{a^4 - a}{d} = \frac{a^4 - a}{a^3 - a} = \frac{a^2 + a + 1}{a + 1} = a + \frac{1}{a+1}$$

is not an integer for any positive integer a, which shows $a^4 \notin \mathcal{A}$.

Conclusion. The sought arithmetic progression is the progression with the first element a and the difference $d = a^3 - a$.

3. *Let $ABCDEF$ be an inscribed hexagon, in which $AB \perp BD$, $BC = EF$. Moreover, suppose that the lines BC and EF cut AD in P, Q, respectively. Denote S the midpoint of AD and K, L the incenters of BPS, EQS. Prove that the angle KLD is right.* (Tomáš Jurík)

Solution. Let k be the circumcircle of the hexagon. Since $AP \perp BD$, k is a Thales circle with diameter AD, thus S, the midpoint of AD, is the center of k as well. We show $\angle KDL = 90°$. There is $\angle KDL = \angle KDS + \angle LDS$. Moreover the triangles

KDS and KBS are congruent according to the *sas* theorem: both of them have SK as a side, SD and SB are both diameters of k, and $\angle BSK = \angle KSD$ because SK is the angle bisector of BSP (Fig. 1). Thus $\angle KDS = \angle KBS$. Since BK is the angle

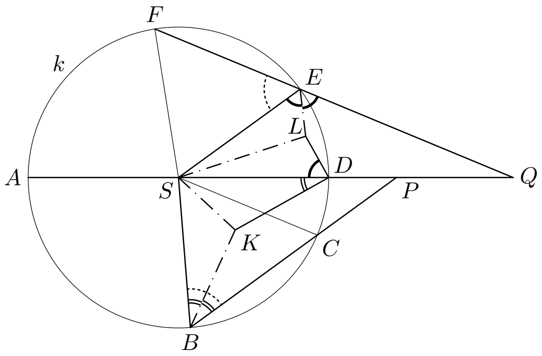

Fig. 1

bisector of SBP we have $\angle KDS = \frac{1}{2}\angle CBS$. Similarly $\angle LDS = \frac{1}{2}\angle QES$.

Now we use the assumption $BC = EF$. Then BCS and EFS are congruent isosceles triangles (with the legs equal to the radius of k), thus $\angle CBS = \angle FES$. In total

$$\angle KDL = \angle KDS + \angle LDS = \frac{1}{2}\angle CBS + \frac{1}{2}\angle QES =$$
$$= \frac{1}{2}\angle FES + \frac{1}{2}\angle QES = \frac{1}{2}(\angle FES + \angle QES) = \frac{1}{2}\cdot 180^\circ = 90^\circ.$$

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4. *Among real numbers a, b, c, and d which satisfy*

$$ab + cd = ac + bd = 4 \quad a \quad ad + bc = 5$$

find these, for which the value of $a+b+c+d$ is the least possible. Find this (the least) value as well. (Jaromír Šimša)

Solution. We have

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2(ab+cd+ac+bd+ad+bc) =$$
$$= a^2+b^2+c^2+d^2+2(4+4+5) = a^2+d^2+b^2+c^2+26. \tag{1}$$

Now $a^2+d^2 \geqslant 2ad$, $b^2+c^2 \geqslant 2bc$ where the equality holds iff $a=d$ and $b=c$ and from (1) we get

$$(a+b+c+d)^2 \geqslant 2ad+2bc+26 = 2\cdot 5+26 = 36.$$

Thus among reals which satisfy the conditions we always have $a+b+c+d \geqslant 6$, where the equality holds iff $a=d$ and $b=c$, or

$$2ab = 4, \quad a^2+b^2 = 5,$$

thus $\{a,b\} = \{1,2\}$.

Conclusion. The sought a, b, c, d are the quadruples $(1,2,2,1)$ and $(2,1,1,2)$ with the least value of $a+b+c+d = 6$.

Final Round of the 61st Czech and Slovak Mathematical Olympiad (March 26–27, 2012)

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1. *Find all integers n such that $n^4 - 3n^2 + 9$ is prime.* (Aleš Kobza)

Solution. We factorize the given expression:

$$n^4 - 3n^2 + 9 = n^4 + 6n^2 + 9 - 9n^2 = (n^2 + 3)^2 - (3n)^2 = (n^2 + 3n + 3)(n^2 - 3n + 3).$$

Should the product be a prime p, one of the factors should be 1 or -1 and the other one then p or $-p$. But both of the factors are only positive (they have negative discriminants), thus the only possibilities are

$$n^2 + 3n + 3 = 1 \qquad \text{or} \qquad n^2 - 3n + 3 = 1.$$

The roots of the first equation are $n = -1$ and $n = -2$, the second equation has roots $n = 1$ and $n = 2$. In both cases the values of the other factor are 7 or 13 which are primes.

Conclusion. The number $n^4 - 3n^2 + 9$ is prime iff $n \in \{-2, -1, 1, 2\}$.

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2. *Find the greatest possible area of a triangle ABC with medians satisfying $t_a \leqslant 2$, $t_b \leqslant 3$, $t_c \leqslant 4$.* (Pavel Novotný)

Solution. Let T be the centroid of ABC and K, L, M be the midpoints of BC, CA, AB. Medians cut ABC into six smaller triangles, each with the same area: for example in the triangle AMT we have $AM = \frac{1}{2}c$, its altitude through T is $\frac{1}{3}v_c$ long, that is $S_{AMT} = \frac{1}{2} \cdot \frac{1}{2}c \cdot \frac{1}{3}v_c = \frac{1}{6} \cdot \frac{1}{2}c \cdot v_c = \frac{1}{6}S_{ABC}$. Analogously for the other triangles.

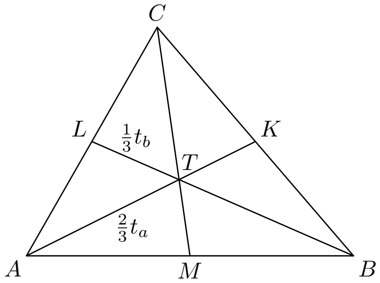

Fig. 1

Thus we will seek for the greatest possible area of one of the triangles, say ATL (Fig. 1), and then we multiply the result by six. There is

$$AT = \frac{2}{3}t_a \leqslant \frac{4}{3}, \qquad TL = \frac{1}{3}t_b \leqslant 1.$$

Therefore we can constrain the area:

$$S_{ATL} = \frac{1}{2}AT \cdot TL \cdot \sin\angle ATL \leqslant \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

Thus the area of the ABC can be at most $6 \cdot \frac{2}{3} = 4$, where the equality holds iff $t_a = 2$, $t_b = 3$ and $\angle ATL = 90°$.

There is a triangle satisfying the conditions with the area 6 indeed: first we construct right triangle ATL with the legs $AT = \frac{4}{3}$ and $TL = 1$. Then C is the symmetric image of A with respect to L and B is the image of L under the homothety with center T and coefficient -2 (Fig. 2). It is easy to count the length of AB. For

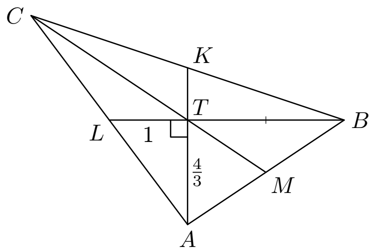

Fig. 2

example the theorem of Pythagoras in ABT gives

$$AB = \sqrt{AT^2 + TB^2} = \sqrt{\frac{16}{9} + 4} = \sqrt{\frac{52}{9}} = \frac{2}{3}\sqrt{13}.$$

Since M is on the Thales' circle with the diameter AB, there is $MT = \frac{1}{2}AB = \frac{1}{3}\sqrt{13}$. Then $t_c = 3 \cdot MT = \sqrt{13} < 4$.

Conclusion. The greatest possible area of the triangle ABC is 4.

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3. *Prove that among any* 101 *real numbers one can choose* u *and* v *such that*

$$100\,|u - v| \cdot |1 - uv| \leqslant (1 + u^2)(1 + v^2).$$

(Pavel Calábek)

Solution. Since $1 + x^2 > 0$ for any real x we get equivalent inequalities:

$$100\,|(u - v)(1 - uv)| \leqslant (1 + u^2)(1 + v^2),$$
$$100\,\left|u - v - u^2v + uv^2\right| \leqslant (1 + u^2)(1 + v^2),$$
$$\left|u(1 + v^2) - v(1 + u^2)\right| \leqslant \frac{1}{100}(1 + u^2)(1 + v^2),$$
$$\left|\frac{u}{1 + u^2} - \frac{v}{1 + v^2}\right| \leqslant \frac{1}{100}. \tag{1}$$

Values of

$$f(x) = \frac{x}{1+x^2}, \quad x \in \mathbb{R},$$

are in $\langle -\frac{1}{2}, \frac{1}{2} \rangle$, since for any real x we have

$$|x| = \sqrt{1 \cdot x^2} \leqslant \frac{1+x^2}{2} \quad \text{hence} \quad \frac{|x|}{1+x^2} \leqslant \frac{1}{2}.$$

Now divide $\langle -\frac{1}{2}, \frac{1}{2} \rangle$ into one hundred intervals of the length $\frac{1}{100}$. According to the Pigeon hole principle, among any 101 real numbers there are u and v such that $f(u)$ and $f(v)$ lie in the same interval, that is $|f(u) - f(v)| \leqslant \frac{1}{100}$, which is exactly the inequality (1), which is equivalent to the original inequality.

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4. *There is a point X inside a parallelogram $ABCD$. Construct a line, which goes through X and divides the parallelogram into two parts, with the greatest possible difference in their areas.* (Vojtech Bálint)

Solution. The sum of the areas of the two parts, into which the line cuts the parallelogram $ABCD$ is constant, their difference will be the greatest iff the smaller area will be the smallest possible. First notice that if X is the center of $ABCD$, than any line through X divides $ABCD$ into two parts of the same area — they are symmetric reflections of each other with respect to X. Thus any line through X is a solution of the given problem.

Generally, let K, L, M, and N be the midpoints of AB, BC, CD, and DA and let S be the center of $ABCD$. First, let as assume X is inside of the parallelogram $AKSN$ (then the symmetric reflection A' of A with respect to X lies inside $ABCD$).

Consider two lines parallel to sides of $ABCD$ going through A'. Denote P and Q their intersections with AB and AD. Then $APA'Q$ is parallelogram with the center X. Thus any line passing through X divides $APA'Q$ into two shapes of the same area. Each of these shapes lies in different parts, into which the line cuts $ABCD$ (Fig. 3a). That is both of the parts of $ABCD$ have the area at least the half of the area of $APA'Q$. Thus the smaller part of $ABCD$ will have the smallest possible area iff it will be inside of $APA'Q$. This will be the case for the line PQ (Fig. 3b).

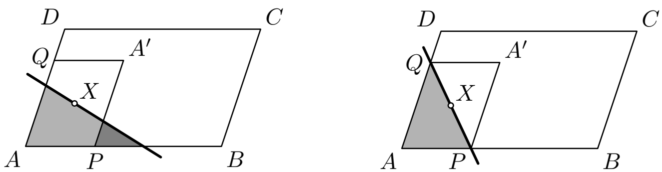

Fig. 3a Fig. 3b

Analogously we find the line, if X is inside of $KBLS$, $SLCM$, or $NSMD$.

Finally if X inside of any of KS, similarly we consider the parallelogram $ABA'B'$ with A' and B' being the symmetric reflections of A and B with respect to X. Now the sought lines again have to cut $ABCD$ into two parts one of which is inside $ABA'B'$. Obviously, such is any line UX, where U is arbitrary point of AB' (Fig. 4a,b).

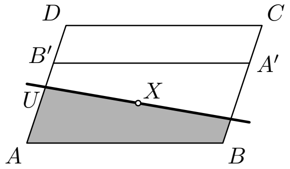

Fig. 4a

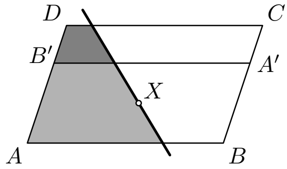

Fig. 4b

Analogously we find the lines if X is inside of SM, NS, or SL.

Conclusion. If X is the center of quadrilateral $ABCD$ then any line passing through X is the solution, if X is off KM and NL then unique line solves the problem. If X is inside KS, SM, NS, or SL then infinitely many lines are the solution. In each of the cases the construction is straight forward considering the facts, which have been mentioned.

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5. *In a group of* 90 *children, each one has at least* 30 *friends (friendship is mutual). Prove that children can be divided into* 3 *groups containing* 30 *children each, such that any child has a friend in his (her) group.* (Ján Mazák)

Solution. There are

$$V = \binom{90}{30} \cdot \binom{60}{30} \cdot \frac{1}{3!},$$

divisions into three groups of 30 children.

We call a division *bad because of* A, if the child A has no friend in his (her) group in the division. We show, that the number Z of bad divisions (i.e. the divisions which do not meet the conditions of the problem) is less than V.

Let Z_A denotes the number of divisions which are bad because of A. If A has n friends in the group altogether, then there exists

$$\binom{89-n}{29}$$

groups of 30 children, containing A and 29 other children, non of whom is a friend with A.

For each such a group, the children left can be divided in

$$\binom{60}{30} \cdot \frac{1}{2}$$

ways into two groups of 30 children. Thus we get the following estimate for the number of divisions, which are bad because of A (taking into account that $n \geqslant 30$, that is $89 - n \geqslant 59$):

$$Z_A = \binom{89-n}{29} \cdot \binom{60}{30} \cdot \frac{1}{2} \leqslant \binom{59}{29} \cdot \binom{60}{30} \cdot \frac{1}{2} \tag{1}$$

The number of all bad divisions is certainly not greater than a sum of all bad divisions for every child individually (a division can be bad because of more children). Since there is 90 children, according to (1) we get

$$Z \leqslant 90 \cdot \binom{59}{29} \cdot \binom{60}{30} \cdot \frac{1}{2}.$$

Thus to prove $Z < V$ it is sufficient to prove

$$90 \cdot \binom{59}{29} \cdot \binom{60}{30} \cdot \frac{1}{2} < \binom{90}{30} \cdot \binom{60}{30} \cdot \frac{1}{3!}, \tag{2}$$

Equivalent modifications of the inequality yields:

$$45 \cdot \binom{59}{29} < \binom{90}{30} \cdot \frac{1}{6},$$

$$6 \cdot 45 \cdot \frac{59!}{29! \cdot 30!} < \frac{90!}{30! \cdot 60!},$$

$$6 \cdot 45 \cdot 59 \cdot 58 \cdot \ldots \cdot 30 < 90 \cdot 89 \cdot \ldots \cdot 61,$$

$$6 \cdot 45 < \frac{90}{59} \cdot \frac{89}{58} \cdot \ldots \cdot \frac{61}{30}. \tag{3}$$

Any of the 30 fraction on the RHS is apparently greater than $1,5$, therefore RHS $>$ $1{,}5^{30} = 2{,}25^{15} > 2^{15} > 270 = 6 \cdot 45$ and we are done.

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6. *In the domain of real numbers solve the following system of equations*

$$\begin{aligned} x^4 + y^2 + 4 &= 5yz, \\ y^4 + z^2 + 4 &= 5zx, \\ z^4 + x^2 + 4 &= 5xy. \end{aligned}$$

(Jaroslav Švrček)

Solution. First we give an estimate of the LHS of the first equation. Consider the obvious inequality $4x^2 \leqslant x^4 + 4$ (it is equivalent to $0 \leqslant (x^2 - 2)^2$), which holds for any real number x with the equality iff $x = \pm\sqrt{2}$. Then

$$4x^2 + y^2 \leqslant x^4 + y^2 + 4 = 5yz.$$

Analogously we get next two inequalities. Thus we have

$$4x^2 + y^2 \leqslant 5yz, \quad 4y^2 + z^2 \leqslant 5zx, \quad 4z^2 + x^2 \leqslant 5xy,$$

Summing up these inequalities we get

$$x^2 + y^2 + z^2 \leqslant xy + yz + zy,$$

which is equivalent to

$$(x - y)^2 + (y - z)^2 + (z - x)^2 \leqslant 0.$$

Thus

$$x = y = z = \sqrt{2} \quad \text{or} \quad x = y = z = -\sqrt{2}.$$

which are indeed solutions.

Conclusion. The solutions of the given system are the triples $\left(\sqrt{2}, \sqrt{2}, \sqrt{2}\right)$ and $\left(-\sqrt{2}, -\sqrt{2}, -\sqrt{2}\right)$.