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2014

63rd Czech and Slovak Mathematical Olympiad

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First Round of the 63rd Czech and Slovak Mathematical Olympiad Problems for the take-home part (October 2013)

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1. *A number n is a product of three (not necessarily distinct) prime numbers. Adding 1 to each of them, after multiplication we get a larger product $n+963$. Determine the original product n.* (Pavel Novotný)

Solution. We look for $n = p \cdot q \cdot r$, with primes $p \leqslant q \leqslant r$ satisfying

$$(p+1)(q+1)(r+1) = pqr + 963. \tag{1}$$

If $p = 2$, the right-hand side of (1) is odd, hence the factors $q+1$, $r+1$ on the left must be odd too. This implies that $p = q = r = 2$, which contradicts to (1). Thus we have proved that $p \geqslant 3$.

Now we will show that $p = 3$. Suppose on the contrary that $3 < p \leqslant q \leqslant r$. Then the right-hand side of (1) is not divisible by 3. The same must be true for the product $(p+1)(q+1)(r+1)$. Consequently, all the primes p, q, r are congruent to 1 modulo 3, and hence $(p+1)(q+1)(r+1) - pqr$ is congruent to $2 \cdot 2 \cdot 2 - 1 \cdot 1 \cdot 1 = 7$, which contradicts to $(p+1)(q+1)(r+1) - pqr = 963$. Therefore, the equality $p = 3$ is established.

Putting $p = 3$ into (1) we get $4(q+1)(r+1) = 3qr + 963$, which can be rewritten as $(q+4)(r+4) = 975$. In view of the prime factorization $975 = 3 \cdot 5^2 \cdot 13$ and inequalities $7 \leqslant q+4 \leqslant r+4$, we conclude that $q + 4 \leqslant \sqrt{975} < 32$ and hence $q + 4 \in \{13, 15, 25\}$. Since q is a prime, it holds that $q = 11$. Then $r + 4 = 65$, and hence $r = 61$ (which is a prime indeed). Consequently, the problem has a unique solution

$$n = 3 \cdot 11 \cdot 61 = 2\,013.$$

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2. *Let x, y and z be any positive real numbers. Prove the inequality*

$$(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big) \leqslant m^2, \quad \textit{where } m = \min\Big(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}, \frac{y}{x}+\frac{z}{y}+\frac{x}{z}\Big).$$

Find when the equality holds. (Jaroslav Švrček, Jaromír Šimša)

Solution. Since the inequality involves the minimum of two positive numbers and since the function $y = x^2$ is increasing on the set $\mathbb{R}^+$, our task is to verify

$$(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big) \leqslant \Big(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\Big)^2 \text{ and } (x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big) \leqslant \Big(\frac{y}{x}+\frac{x}{z}+\frac{z}{y}\Big)^2, \tag{1}$$

as well as to find when *at least one* equality in (1) holds. Replacing a triple (x,y,z) by the triple (y,x,z), we get the second inequality in (1) from the first one. Thus we can restrict to the proof of the first inequality. Distributing both sides leads to

$$3+\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+\frac{y}{z}+\frac{z}{x}+\frac{z}{y} \leqslant \frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}+2\Big(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\Big).$$

Let us introduce the new (positive) variables $a=x/y$, $b=y/z$, $c=z/x$ and rewrite the last inequality as

$$\Big(a^2-1-a+\frac{1}{a}\Big)+\Big(b^2-1-b+\frac{1}{b}\Big)+\Big(c^2-1-c+\frac{1}{c}\Big) \geqslant 0. \tag{2}$$

For any positive t, we notice that

$$t^2-1-t+\frac{1}{t}=(t^2-1)-\frac{t^2-1}{t}=\frac{(t^2-1)(t-1)}{t}=\frac{(t-1)^2(t+1)}{t}.$$

This implies that (2) holds as well and that (2) becomes an equality if and only if $a=b=c=1$, i.e. $x=y=z$ for the original variables. Note that the last condition does not change under transformation $(x,y,z)\to(y,x,z)$. Thus the original inequality is proven and becomes an equality if and only $x=y=z$.

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3. *We are given a triangle ABC with incentre I. Suppose that there exists an intersection point M of the line AB and the perpendicular to CI through I. Prove that the circumcirle of the triangle ABC intersects the segment CM in an interior point N and that $NI \perp MC$.* (Peter Novotný)

Solution. First we show in two different ways that the line MI, a perpendicular to CI through I, is tangent the to circle ABI. The first way is based on the known fact that AIC a BIC are obtuse angles of measures $90°+\frac{1}{2}\beta$ and $90°+\frac{1}{2}\alpha$, respectively (in common notation for interior angles of $\triangle ABC$). This fact implies that the line MI forms acute angles $\frac{1}{2}\beta$ and $\frac{1}{2}\alpha$ with the segments AI and BI, respectively[1] hence the angles congruent with angles IBA and IAB in circle ABI (Fig. 1). Well known properties of inscribed and subtended angles lead to the conclusion that the line MI is tangent to the circle ABI. The second reason for this conclusion is the based on the known fact that the centre of ABI is the midpoint of the arc AB of circle ABC which lies on the ray CI bisecting $\angle ACB$).

[1] As an consequence we can see that the assumed existence of the intersection point M is equivalent to the inequality $\frac{1}{2}\alpha \neq \frac{1}{2}\beta$ or $\alpha \neq \beta$. Due to the symmetry we can assume that $\alpha > \beta$ as in our figure; the point M then lies on the ray opposite to ray AB and satisfies $|\angle IMA| = \frac{1}{2}\alpha - \frac{1}{2}\beta$.

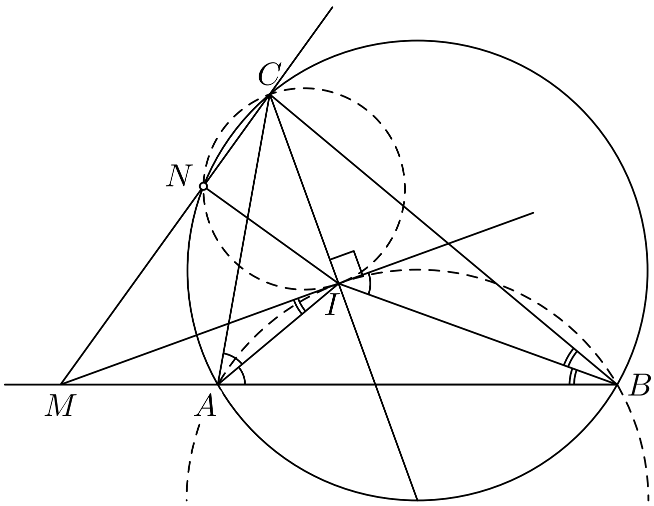

Fig. 1

From the proved tangency of MI to ABI it follows that the point M lies on the line AB outside of the segment AB. Moreover, the power m of M with respect to ABI is positive and given by $m = |MI|^2 = |MA| \cdot |MB|$. Hence M lies in the exterior of the circle ABC (as AB is its chord) and the power of M with respect to ABC is the same $m = |MA| \cdot |MB|$. Since $m = |MI|^2 < |MC|^2$ from the right-angled triangle CMI, it holds that $|MA| \cdot |MB| < |MC|^2$. This means that the circle ABC intersects the segment MC in an interior point N, because $|MN| \cdot |MC| = |MA| \cdot |MB|$ implies that $|MN| < |MC|$ for the second point N of intersection of the ray MC with ABC. This proves the first conclusion of the problem.

To show that CNI is a right angle, we use the proved equality $|MC| \cdot |MN| = |MI|^2$ and apply a familiar theorem to the leg MI of the right-angled triangle CMI: Its altitude from the vertex I meets the hypotenuse CM in such a point X which is determined by equation $|MC| \cdot |MX| = |MI|^2$. Thus we have $X = N$ in our case and the solution is complete.

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4. *Let $l(n)$ denote the the greatest odd divisor of any natural number n. Find the sum*

$$l(1) + l(2) + l(3) + \cdots + l\big(2^{2013}\big).$$

(Michal Rolínek)

Solution. For each natural k, the equalities $l(2k) = l(k)$ and $l(2k-1) = 2k-1$ are clearly valid. Thus we can add the values $l(n)$ over groups of numbers n lying always between two consecutive powers of 2. In this way we will prove by induction the formula

$$s(n) = l\big(2^{n-1}+1\big) + l\big(2^{n-1}+2\big) + l\big(2^{n-1}+3\big) + \cdots + l\big(2^n\big), \qquad (1)$$

for $n = 1, 2, 3, \ldots$ The case $n = 1$ is trivial. If $s(n) = 4^{n-1}$ pro some n, then

$$\begin{aligned} s(n+1) &= l\big(2^n+1\big) + l\big(2^n+2\big) + l\big(2^{n-1}+3\big) + \cdots + l\big(2^{n+1}\big) \\ &= \big[(2^n+1) + (2^{n-1}+3\big) + \cdots + \big(2^{n+1}-1)\big] + s(n) \\ &= \frac{2^{n-1}}{2}\big(2^n + 1 + 2^{n+1} - 1\big) + 4^{n-1} = 2^{n-2} \cdot 3 \cdot 2^n + 4^{n-1} = 4^n. \end{aligned}$$

(We have used the fact the number of all odd numbers from $2^n + 1$ to $2^{n+1} - 1$ [including both limits] equals 2^{n-1}.) The proof of (1) by induction is complete.

Using the formula (1) we compute the requested sum as follows:

$$\begin{aligned} l(1)+&l(2)+l(3)+\cdots+l\left(2^{2013}\right)=l(1)+s(2)+s(3)+\cdots+s(2\,013) \\ &=1+1+4+4^2+4^3+\cdots+4^{2012}=1+\frac{4^{2013}-1}{3}=\frac{4^{2013}+2}{3}. \end{aligned}$$

Remark. It is worth mentioning that the formula $s(n) = 4^{n-1}$ is a special case of a more general (and surprising) formula

$$l(k+1)+l(k+2)+l(k+3)+\cdots+l(2k)=k^2, \tag{2}$$

which can be proved itself for each natural number k even without using induction. Indeed, all the k summands on the left-hand side of (2) are obviously numbers of the k-element subset $\{1, 3, 5, \ldots, 2k-1\}$, and moreover, these summands are pairwise distinct, because the ratio of any two numbers from $\{k+1, k+2, \ldots, 2k\}$ is not a power of 2. Consequently, the sum in (2) equals the sum $1 + 3 + 5 + \cdots + (2k - 1)$, which is k^2 as stated in (2).

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5. *Determine the number of all coverings of a chessboard* 3×10 *by (nonoverlapping) pieces* 2×1 *which can be placed both horizontally and vertically.*

(Stanislava Sojáková)

Solution. Let us solve a more general problem of determining the number a_n of all coverings of a chessboard $3 \times 2n$ by pieces 2×1, for a given natural n.[2] We will attack the problem by a recursive method, starting with $n = 1$.

The value $a_1 = 3$ (for the chessboard 3×3) is evident (see Fig. 2). To prove that $a_2 = 11$ by a direct drawing all possibilities is too laborious. Instead of this, we introduce new numbers b_n: Let each b_n denote the number of all "incomplete" coverings of a chessboard $3 \times (2n - 1)$ by $3n - 2$ pieces 2×1, when a fixed corner field 1×1 (specified in advance, say the lower right one) remains uncovered. Thanks to the axial symmetry, the numbers b_n remain to be the same if the fixed uncovered corner field will be the upper right one. Moreover, it is clear that $b_1 = 1$.

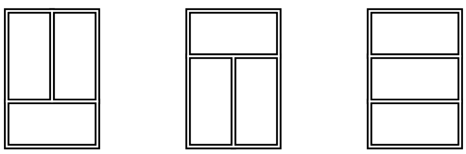

Fig. 2

Now we are going to prove that for each $n > 1$, the following equalities hold:

$$b_n = a_{n-1} + b_{n-1} \quad \text{and} \quad a_n = a_{n-1} + 2b_n. \tag{1}$$

[2] For an obvious reason, we consider a chessboard $3 \times k$ with an even k only.

The first equality in (1) follows from a partition of all (above described) “incomplete” coverings of a chessboard $3 \times (2n-1)$ into two (disjoint) classes which are formed by coverings of types A and B, respectively, see Fig. 3. Notice that the numbers of elements (i.e. coverings) in the two classes are a_{n-1} and b_{n-1}, respectively.

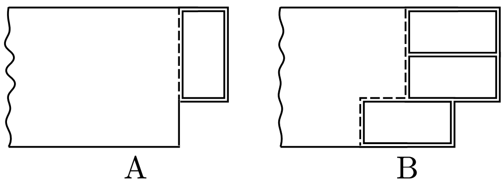

Fig. 3

Similarly, the second equality in (1) follows from a partition of all coverings of a chessboard $3 \times 2n$ into three (disjoint) classes which are formed by coverings of types C, D and E respectively, see Fig. 4. It is evident that the numbers of elements in the three classes are a_{n-1}, b_n and b_n, respectively.

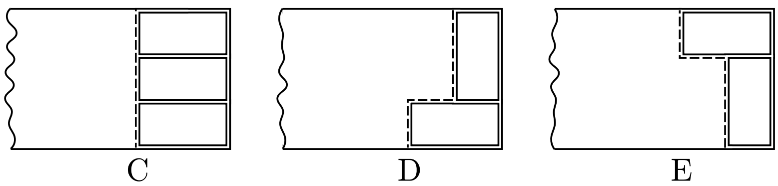

Fig. 4

Now we are ready to compute the requested number a_5. Since $a_1 = 3$ and $b_1 = 1$, the proved equalities (1) successively yield

$$b_2 = a_1 + b_1 = 4,\ a_2 = a_1 + 2b_2 = 11,\ b_3 = a_2 + b_2 = 15,\ a_3 = a_2 + 2b_3 = 41,$$
$$b_4 = a_3 + b_3 = 56,\ a_4 = a_3 + 2b_4 = 153,\ b_5 = a_4 + b_4 = 209,\ a_5 = a_4 + 2b_5 = 571.$$

Answer. The number of coverings of the chessboard 3×10 equals 571.

Remark. Let us show that the numbers a_n of coverings of a chessboard $3 \times 2n$ by pieces 2×1 satisfy the following recurrence equation

$$a_{n+2} = 4a_{n+1} - a_n \quad \text{for each } n \geqslant 1. \tag{2}$$

(Thus the numbers a_n can be computed without using auxiliary numbers b_n from the above solution.) We prove (2) by excluding the numbers from the relations (1):

$$\begin{aligned} a_{n+2} &= a_{n+1} + 2b_{n+2} = a_{n+1} + 2(a_{n+1} + b_{n+1}) \\ &= 3a_{n+1} + 2b_{n+1} = 3a_{n+1} + (a_{n+1} - a_n) = 4a_{n+1} - a_n. \end{aligned}$$

Finally, let us remind a well known result: each sequence $(a_n)_{n=1}^{\infty}$ of numbers satisfying (2) is of the form $a_n = C_1\lambda_1^n + C_2\lambda_2^n$, where $\lambda_{1,2} = 2 \pm \sqrt{3}$ are the roots of the equation $\lambda^2 = 4\lambda - 1$ while $C_{1,2}$ are arbitrary constants. Taking in account our values $a_1 = 3$ and $a_2 = 11$, we conclude that for each n, the number a_n of all coverings of a chessboard $3 \times 2n$ by pieces 2×1 is given by a direct formula

$$a_n = \frac{3+\sqrt{3}}{6} \cdot \left(2+\sqrt{3}\right)^n + \frac{3-\sqrt{3}}{6} \cdot \left(2-\sqrt{3}\right)^n.$$

6. *We are given a triangle ABC. Find the locus of points X in the plane ABC whose reflections through the lines AB, BC, CA are vertices of an equilateral triangle.* (Pavel Calábek)

Solution. For any point X of the plane ABC, let X_a, X_b and X_c denote the reflections of X through the lines BC, CA and AB, respectively (Fig. 5). First we prove that the distances between any two of the points X_a, X_b and X_c are given in general by formulæ

$$|X_aX_b| = 2|XC|\sin\gamma, \quad |X_aX_c| = 2|XB|\sin\beta, \quad |X_bX_c| = 2|XA|\sin\alpha, \tag{1}$$

in which α, β, γ denote the interior angles of the triangle ABC as usual.

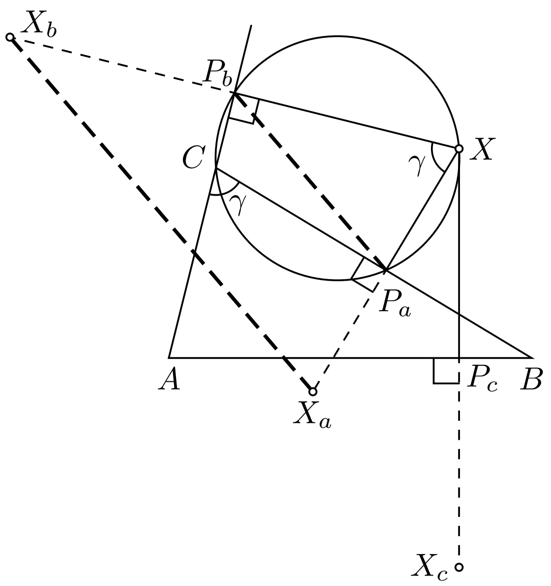

Fig. 5

It suffices to prove the first equality (1) which is obvious if $X = C$, because then $X_a = X_b$ $(= X)$. If $X \neq C$, then the segment XC is a diameter of a circle (see Fig. 5) which passes through the marked orthogonal projections P_a and P_b of X onto BC and CA, respectively (Thales' theorem). Since the chord P_aP_b subtends inscribed angles γ and $180° - \gamma$, Law of Sines implies that $|P_aP_b| = |XC|\sin\gamma$. Using the homothety with centre X and ratio 2, we conclude that $|X_aX_b| = 2|P_aP_b|$, and hence the equalities (1) are established for any point X.

The proved formulæ (1) imply that our task is to find exactly such points X in the plane ABC that satisfy

$$2|XA|\sin\alpha = 2|XB|\sin\beta = 2|XC|\sin\gamma > 0$$

(recall that the triangle $X_aX_bX_c$ has to be equilateral). Otherwise speaking, we look for all points X whose distances to A, B and C are positive and proportional as follows:

$$|XA| : |XB| : |XC| = \frac{1}{\sin\alpha} : \frac{1}{\sin\beta} : \frac{1}{\sin\gamma} = \frac{1}{|BC|} : \frac{1}{|AC|} : \frac{1}{|AB|}$$

(we have turned from angles to sides of $\triangle ABC$ using Law of Sines again). Such points X are determined as common points of the following three *circles of Apollonius* (i.e. sets of points in the plane which have a specified ratio of distances to two fixed points):

$$k_a\colon \frac{|XB|}{|XC|} = \frac{|AB|}{|AC|}, \quad k_b\colon \frac{|XA|}{|XC|} = \frac{|AB|}{|BC|}, \quad k_c\colon \frac{|XA|}{|XB|} = \frac{|AC|}{|BC|}. \tag{2}$$

It is clear that any point shared by two of the circles lies on the third circle as well.[3] It follows from (2) that $A \in k_a$, $B \in k_b$ a $C \in k_c$, which simplifies the construction of the three circles in practice: If the bisectors of interior angles in $\triangle ABC$ cut its interior in segments AK, BL and CM (Fig. 6), then $K \in k_a$, $L \in k_b$ and $M \in k_c$ (an immediate consequence of the well known proportions such as $|KB| : |KC| = |AB| : |AC|$). Hence the centre of k_a can be constructed as the intersection point of the line BC and the perpendicular bisector of the segment AK (excluding the case $|AB| = |AC|$, when k_a becomes simply the perpendicular bisector of BC). Similarly, using the perpendicular bisectors of BL and CM we get centres od k_b and k_c, respectively.)

Figure 6 illustrates the case when the circles k_a, k_b, k_c meet in two distinct points and hence the given problem has two solutions marked as X and Y, with the corresponding equilateral triangles $X_aX_bX_c$ and $Y_aY_bY_c$, respectively.[4]

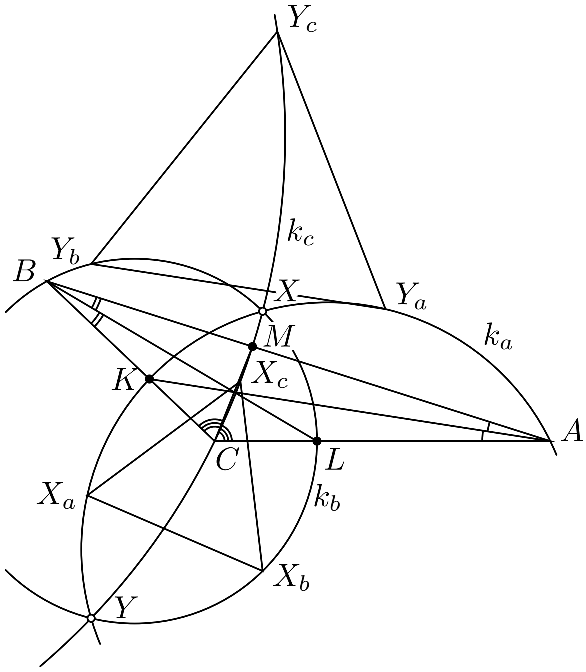

Fig. 6

Despite of the fact that the requested locus of points X is determined (by an Euclidean construction), we have to discuss how the number of solutions depends on

[3] Some of these three sets (one or three) can be straight lines instead of circles — if the corresponding ratio equals 1. We postpone this question to the closing discussion.

[4] Afterwards we prove that two solutions exist whenever the given triangle ABC is not equilateral. The fact that the last is not a trivial conclusion is supported by an observation that both solutions X and Y in Fig. 6 are situated in the exterior of the triangle ABC.

the choice of $\triangle ABC$. As we know, this reduces to the question of common points of any two of the circles k_a, k_b, k_c. This matter becomes easier if we involve again into consideration the segments AK, BL, CM from Fig. 6 which are chords of k_a, k_b and k_c, respectively.

Discussion.

a) If the triangle ABC is *equilateral*, the “circles” k_a, k_b, k_b are in fact perpendicular bisectors of the sides of $\triangle ABC$. Consequently, the problem has a unique solution — a point X which coincides with the incentre of $\triangle ABC$.

b) If the triangle ABC is *isosceles* (but not equilateral), say if $|AB| \neq |AC| = |BC|$, then the circle k_c is a perpendicular bisector of the base AB which meets the circle k_a in two points, because k_c meets the interior of the chord AK, and hence the both arcs AK of the circle k_a as well. Consequently, the problem has two solutions.

c) Suppose that the triangle ABC is *scalene*, with the largest side, say AB (as in Fig. 6). Then the ratio $|XB|/|XC|$ for points $X \in k_a$ is larger than 1, because of $A \in k_a$. Hence B lies in the interior k_a, while C lies in its exterior. The last together with $A \in k_a$ implies that L, an interior point of AC, lies in the exterior of k_a. Thus k_a intersects the chord BL of k_b which means that k_a and k_b meet in two points. Consequently, the problem has two solutions.

First Round of the 63rd Czech and Slovak Mathematical Olympiad (December 10th, 2013)

1. *Prove that for each integer number n, $n \geqslant 3$, the following $2n$-digit number*

$$\underbrace{1\ldots1}_{n-1}2\underbrace{8\ldots8}_{n-2}96$$

is a perfect square. (Vojtech Bálint)

Solution. The number under consideration can be expressed as follows:

$$\begin{aligned}
&\left(10^{2n-1}+10^{2n-2}+\cdots+10^{n+1}\right)+2\cdot10^{n}+8\cdot\left(10^{n-1}+10^{n-2}+\cdots+10^{2}\right)+96\\
&\quad=10^{n+1}\cdot\frac{10^{n-1}-1}{9}+2\cdot10^{n}+8\cdot10^{2}\cdot\frac{10^{n-2}-1}{9}+96\\
&\quad=\frac{10^{2n}-10^{n+1}+18\cdot10^{n}+800\cdot10^{n-2}-800+9\cdot96}{9}\\
&\quad=\frac{10^{2n}+16\cdot10^{n}+64}{9}=\left(\frac{10^{n}+8}{3}\right)^{2}.
\end{aligned}$$

As required, we have obtained a perfect square, because the number 10^n+8 is divisible by 3, as the sum of its digits equals 9.

Another solution. Starting with examples

$$1\,296=36^2,\ 112\,896=336^2,\ 11\,128\,896=3\,336^2,\ \ldots,$$

we easily guess that for each $n \geqslant 2$,

$$\underbrace{1\ldots1}_{n-1}2\underbrace{8\ldots8}_{n-2}96=\underbrace{33\ldots3}_{n-1}6^2.$$

The exact proof can be done by using the usual multiplication scheme:

$$\begin{array}{r}
333\ldots3336\\
\times333\ldots3336\\
\hline
2000\ldots0016\\
10000\ldots008\\
100000\ldots08\\
1000000\ldots8\\
\vdots\qquad\qquad\vdots\\
1\ldots000008\\
10\ldots00008\\
100\ldots0008\\
\hline
111\ldots112888\ldots8896
\end{array}$$

Both (identical) factors are n-digit, hence an $(n+1)$-digit number stands in each of the n rows between the two delimiting lines. From this fact, it is easy to determine the values of digits (including the numbers of appearances) in the resulting product.

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2. *Let M be the midpoint of the side AB of a triangle ABC. Prove that the equality $|\angle ABC| + |\angle ACM| = 90°$ holds if and only if the triangle ABC is isosceles or right-angled, with AB as a base or a hypotenuse, respectively.* (Pavel Novotný)

Solution. Assume first that $|\angle ABC| + |\angle ACM| = 90°$. Using the notation $\phi = |\angle ACM|$ and $\psi = |\angle BCM|$ (Fig. 1), we conclude from our assumption that $|\angle ABC| = 90° - \phi$, and hence $|\angle BAC| = 90° - \psi$ as well, because of an easy angle computation in $\triangle ABC$:

$$\begin{aligned}|\angle BAC| &= 180° - |\angle ABC| - |\angle ACB| \\ &= 180° - (90° - \phi) - (\phi + \psi) = 90° - \psi.\end{aligned}$$

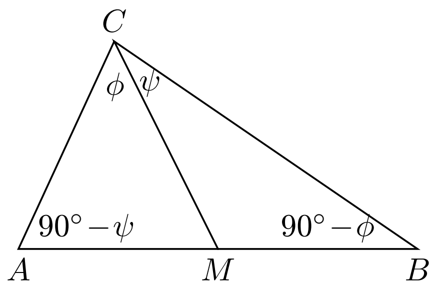

Fig. 1

Applying Law of Sines to $\triangle ACM$ and $\triangle BCM$, we get

$$\frac{\sin(90° - \psi)}{\sin\phi} = \frac{|CM|}{|AM|} = \frac{|CM|}{|BM|} = \frac{\sin(90° - \phi)}{\sin\psi}.$$

Comparing the two ratios of sines and using the formula $\sin(90° - \omega) = \cos\omega$, we obtain an equality $\sin\phi\cos\phi = \sin\psi\cos\psi$ or $\sin 2\phi = \sin 2\psi$. Since the angles ϕ and ψ are acute, both 2ϕ and 2ψ are between $0°$ and $180°$. Thus by a well known sine property, the equality $\sin 2\phi = \sin 2\psi$ means that either $2\phi = 2\psi$ or $2\phi + 2\psi = 180°$. In the first case (when $\phi = \psi$), the interior angle of $\triangle ABC$ at the vertices A and B are equal, in the second case (when $\phi + \psi = 90°$) the interior angle at the vertex C is right. This completes the proof of one of the two implications stated in the problem.

To prove the second (converse) implication, let us assume that (i) $|AC| = |BC|$ or (ii) $|\angle ACB| = 90°$.

Case (i). It follows from $|AC| = |BC|$ that the triangles ACM and BCM are congruent (by SSS theorem), with right interior angles at the vertex M. Consequently,

$$|\angle ABC| + |\angle ACM| = |\angle MBC| + |\angle BCM| = 180° - |\angle BMC| = 90°.$$

Case (ii). It follows from $|\angle ACB| = 90°$ that $|MB| = |MC|$ by Thales' theorem. Thus the angles MCB and MBC (or ABC) are congruent and hence

$$|\angle ABC| + |\angle ACM| = |\angle MCB| + |\angle ACM| = |\angle ACB| = 90°.$$

The converse implication is proven.

Another solution. Let k be the circumcircle of the given triangle ABC. Its median CM can be extended to the chord CC' of the circle k (Fig. 2). Since the inscribed angles ABC' and ACC' (or ACM) are congruent, the considered sum of angles ABC and ACM is equal to the angle CBC'. By Thales' theorem the last angle CBC' is right if and only if the chord CC' is a diameter of the circle k. This happens if and only if the centre S of k lies on the ray CM. For such a situation, we distinguish two cases: $S = M$ and $S \neq M$. Note that $S = M$ holds if and only if the angle ACB is right (by Thales' theorem again). Thus let us analyse the second case $S \neq M$: The three distinct point C, M and S are obviously collinear if and only if the line MS, a perpendicular bisector of a segment AB, passes through the point C. However, the last condition is equivalent to the desired equality $|AC| = |BC|$. This completes the proof (common for the both composing implications).

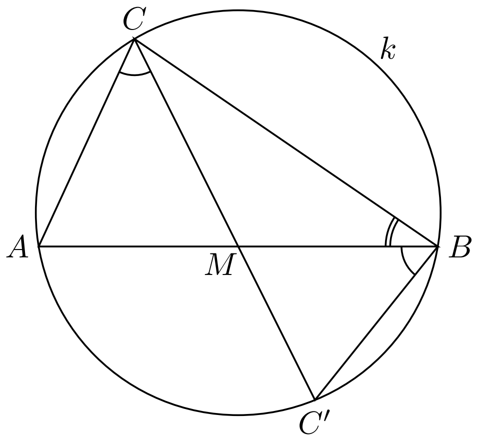

Fig. 2

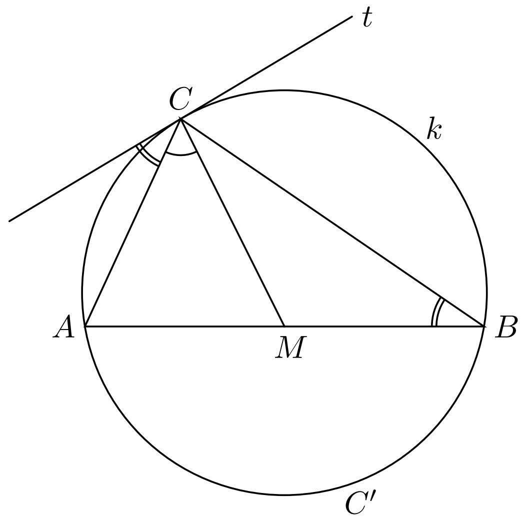

Fig. 3

Remark. Instead of the chord CC' of the circumcircle k, it is possible to consider the tangent line t to the circle k at its point C (Fig. 3). Since the inscribed angle ABC is always congruent to the marked angle between AC and t, the sum of the angles ABC a ACM equals $90°$ if and only if the tangent t is perpendicular to the ray CM. The last is equivalent to the condition from the above solution, namely that the ray CM passes through the centre S of k.

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3. *We are given a sheet of paper in the form of a rectangle $x \times y$, where x and y are integer numbers larger than 1. Let us draw a lattice of $x \cdot y$ unite squares on the sheet. Rolling up the rectangle and gluing it along its opposite sides we shape a lateral surface of a circular cylinder. Join each two distinct vertices of the marked unit squares on the surface by a segment. How many of all these segments are passing through an interior point of the cylinder? In the case $x > y$ decide when this number of "internal" segments is larger — for the cylinder with bases of perimeter x, or y?* (Vojtech Bálint)

Solution. We will compute the requested number P of all internal segments for the cylinder formed by gluing the rectangle $x \times y$ along the opposite sides of length y.

This cylinder has two bases of perimeter x and its lateral sides are of length y. We will use an obvious formula $P = P_0 - P_1 - P_2$, where P_0 denotes the total number of segments, while P_1 and P_2 denote the numbers of segments which lie on the lateral surface or on one of the two bases, respectively.

Note that the vertices of the unit squares are situated on the surface of the cylinder in such a way that exactly $y+1$ of them lie on the same of the x lateral sides and, in the same time, exactly x vertices lie on the same boundary circle of the two bases. These facts lead immediately to the following formulæ

$$P_0 = \binom{x(y+1)}{2} = \frac{x(y+1)(xy+x-1)}{2},$$
$$P_1 = x \cdot \binom{y+1}{2} = \frac{x(y+1)y}{2},$$
$$P_2 = 2 \cdot \binom{x}{2} = x(x-1).$$

Consequently,

$$\begin{aligned} P = P_0 - P_1 - P_2 &= \frac{x(y+1)(xy+x-1)}{2} - \frac{x(y+1)y}{2} - x(x-1) \\ &= \frac{x(x-1)(y^2+2y-1)}{2}. \end{aligned}$$

In view of symmetry, the number Q of internal segments for the other cylinder (with base's perimeter y and lateral sides of length x) is given by

$$Q = \frac{y(y-1)(x^2+2x-1)}{2}.$$

To decide which of the inequalities $P > Q$ or $Q > P$ holds in the case when $x > y$, we factorize the difference $P - Q$ (since $P = Q$ if $x = y$, the polynomial $P - Q$ must be divisible by $x - y$):

$$\begin{aligned} 2(P-Q) &= (x^2-x)(y^2+2y-1) - (y^2-y)(x^2+2x-1) \\ &= (x^2y^2 - xy^2 + 2x^2y - 2xy - x^2 + x) \\ &\quad - (x^2y^2 - x^2y + 2xy^2 - 2xy - y^2 + y) \\ &= 3xy(x-y) - (x-y)(x+y) + (x-y) \\ &= (x-y)(3xy - x - y + 1). \end{aligned}$$

Thus $x > y$ implies that $P > Q$ if we show that the same condition $x > y$ implies that $3xy - x - y + 1 > 0$. The last is almost evident: it follows from $y \geqslant 2$ that $3xy \geqslant 6x$ and hence

$$3xy - x - y + 1 \geqslant 5x - y + 1 > 4x + 1 > 0.$$

Answer. In the case when $x > y$, the number of internal segments is larger for the cylinder with bases which perimeter has length x.

Remark. Let us describe a shorter way of determining the number P. The orthogonal projection of each internal segment to the fixed base of the cylinder is one of the $\frac{1}{2}x(x-1)$ segments connecting x vertices on the boundary circle. Each of these projections is common for exactly $(y+1)^2-2=y^2+2y-1$ internal segments, because $y+1$ is the number of vertices on the same lateral side and all the segments connecting two distinct lateral sides are internal, with exception of the two segments lying on the bases of the cylinder. Hence,

$$P=\frac{x(x-1)(y^2+2y-1)}{2}.$$

Second Round of the 63rd Czech and Slovak Mathematical Olympiad (January 14th, 2014)

1. *Find all positive integers n which are not powers of 2 and which satisfy the equation $n = 3D + 5d$, where D (and d) denote the greatest (and the least) numbers among the all odd divisors of n which are larger than 1.*

(Tomáš Jurík)

Solution. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ be the prime factorization of a satisfactory number n. Here $p_1 < p_2 < \dots < p_k$ are all the prime divisors of n and the exponents α_i are positive integers. The given equation implies that $p_1 = 2$ (otherwise $D = n$ which contradicts to $n = 3D + 5d$) and that $k \geqslant 2$ (otherwise n is a power of 2). Thus we have $D = p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $d = p_2$ and the equation becomes

$$2^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = 3p_2^{\alpha_2} \dots p_k^{\alpha_k} + 5p_2 \quad \text{or} \quad \bigl(2^{\alpha_1} - 3\bigr) p_2^{\alpha_2 - 1} \dots p_k^{\alpha_k} = 5.$$

(In the case when $k = 2$ the left-hand of the last equation is simply $\bigl(2^{\alpha_1} - 3\bigr) p_2^{\alpha_2 - 1}$.) Since the number 5 has only two divisors 1 and 5, it holds that $2^{\alpha_1} - 3 \in \{1, 5\}$ and hence either $\alpha_1 = 2$ or $\alpha_1 = 3$.

(i) The case $\alpha_1 = 2$. The simplified equation

$$p_2^{\alpha_2 - 1} \dots p_k^{\alpha_k} = 5$$

holds if and only if either $k = 2$, $p_2 = 5$ and $\alpha_2 - 1 = 1$, or $k = 3$, $\alpha_2 - 1 = 0$, $p_3 = 5$ and $\alpha_3 = 1$ — then from $2 < p_2 < p_3 = 5$ it follows that $p_2 = 3$. Consequently, there are exactly two solutions in the case (i), namely $n = 2^2 5^2 = 100$ and $n = 2^2 3^1 5^1 = 60$.

(ii) The case $\alpha_1 = 3$. The simplified equation

$$p_2^{\alpha_2 - 1} \dots p_k^{\alpha_k} = 1$$

holds only for $k = 2$ and $\alpha_2 - 1 = 0$. Notice that there is no restriction on the prime number p_2 excepting the inequality $p_2 > 2$. Consequently, there are infinitely many solutions in the case (ii) and all of them are given by $n = 2^3 p_2^1 = 8p_2$, where p_2 is any odd prime number.

Answer. All the solutions n are: $n = 60$, $n = 100$ and $n = 8p$, where p is any odd prime number.

Remark. Let us show that the above solution can be presented more simply, if the “full” prime factorization of n is replaced by a “partial” factorization $n = 2^\alpha pl$, in which 2^α is the greatest power of 2 dividing n, p is the smallest prime divisor of n and and l is and odd number which has no prime divisor smaller than p (thus we have either $l = 1$, or $l \geqslant p$). Using this factorization, we can write $D = pl$, $d = p$ and hence our task is to solve the equation

$$n = 2^\alpha pl = 3pl + 5p \quad \text{or} \quad \bigl(2^\alpha - 3\bigr)l = 5.$$

It follows that either $l = 1$ and $2^\alpha - 3 = 5$, or $l = 5$ and $2^\alpha - 3 = 1$. In the first case we have $\alpha = 3$ and thus $n = 8p$, where p is any prime number. In the second case it holds that $l = 5$, $\alpha = 2$ and thus $n = 20p$, but $5 = l \geqslant p$ implies that $p \in \{3, 5\}$, so there are only two solutions $n \in \{60, 100\}$.

Another solution. The given equation $n = 3D + 5d$ implies that $n > 3D$ and $n \leqslant 3D + 5D = 8D$ (because of $d \leqslant D$). Since the ratio $n : D$ must be a power of 2, it follows from $3 < n : D \leqslant 8$ that either (i) $n = 4D$, or (ii) $n = 8D$.

(i) The case $n = 4D$. From $4D = n = 3D + 5d$ we have $D = 5d$ and thus $n = 4D = 20d$. Since d must be a prime odd divisor of n which is a multiple of 5, we conclude that $p \in \{3, 5\}$ and hence $n \in \{60, 100\}$ (both the values are clearly satisfactory).

(ii) $n = 8D$. Our way of deriving the inequality $n \leqslant 8D$ implies now that $D = d$ a hence D is an (odd) prime number. All such $n = 8D$ are solutions indeed.

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2. *We are given two circles $k_1(S_1, r_1)$ and $k_2(S_2, r_2)$ in the plane, with $|S_1S_2| > r_1 + r_2$. Find the locus of points X which do not lie on the line S_1S_2 and possess the following property: The segments S_1X and S_2X intersect successively the circles k_1 and k_2 in such points whose distances to the line S_1S_2 are the same.*
(Jaromír Šimša)

Solution. In the first part of our solution, we will assume that X is any point with the required property. It is clear that X lies in the exteriors of the circles k_1 and k_2 and that the points S_1, S_2 and X are vertices of a triangle whose sides S_1X, S_2X are intersected successively by the circles k_1 and k_2 in such points Y_1 a Y_2 which lie on the same line parallel to S_1S_2 (Fig. 1). Since the triangles XY_1Y_2 and XS_1S_2 are similar (by theorem AA), it holds

$$\frac{|XY_1|}{|XS_1|} = \frac{|XY_2|}{|XS_2|}, \tag{1}$$

which can be rewritten, because of the equalities

$$|XY_1| = |XS_1| - r_1 \quad \text{a} \quad |XY_2| = |XS_2| - r_2, \tag{2}$$

as an equation for lengths of the segments XS_1 and XS_2:

$$\frac{|XS_1| - r_1}{|XS_1|} = \frac{|XS_2| - r_2}{|XS_2|},$$

or

$$\frac{|XS_1|}{|XS_2|} = \frac{r_1}{r_2}. \tag{3}$$

Since the points S_1 and S_2 are fixed as well as the ratio r_1/r_2, the locus of points X

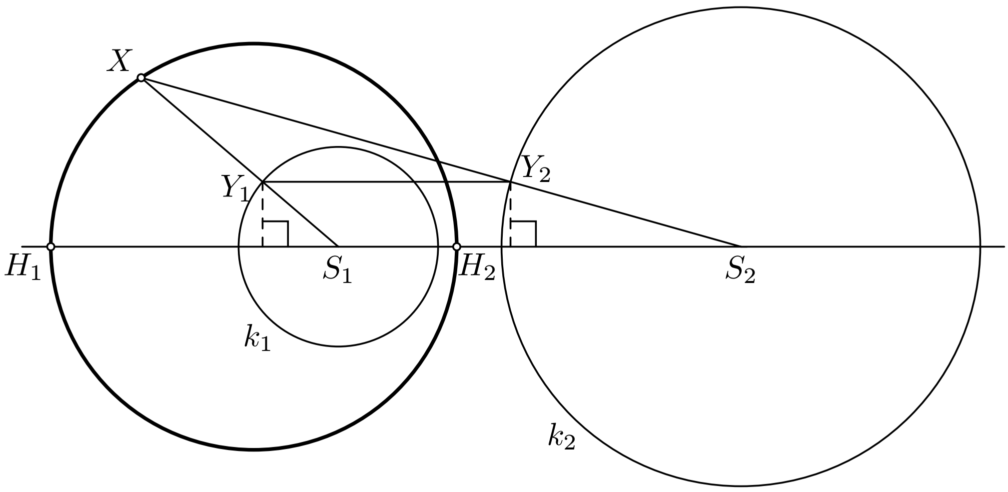

Fig. 1

satisfying equation (3) is a circle of Apollonius (which evidently becomes a straight line if the ratio r_1/r_2 equals 1). As known for the case when $r_1/r_2 \neq 1$, the are exactly two solutions $X = H_1$ and $X = H_2$ of the equation (3) that lie on the line S_1S_2 and form a diameter H_1H_2 of the resulting circle of Apollonius. For our situation, let us add the fact that the points H_1 and H_2 are centres of both homotheties of the initially given circles k_1 and k_2.

In the second part of our solution, we will conversely assume that X is a point of the circle of Apollonius given by the equation (3) and that X lies outside of the line S_1S_2, i.e. $X \neq H_1$ and $X \neq H_2$. In view of the condition that $|S_1S_2| > r_1 + r_2$, the whole circle of Apollonius (with diameter H_1H_2) lies in the exteriors of the circles k_1 and k_2. Indeed, the last fact follows from a known position of the homothety centres H_1 and H_2 on the line S_1S_2: If for example $r_1 < r_2$, then the diameter H_1H_2 contains the diameter of k_1, while the diameter of k_2 and the diameter H_1H_2 are disjoint. (See also *Remark* below.)

The proven property implies that XS_1S_2 is a triangle with $|XS_1| > r_1$ and $|XS_2| > r_2$. Thus there are points $Y_1 \in k_1$ and $Y_2 \in k_2$ lying on the segments S_1X and S_2X, respectively. Since the equalities (2) are valid again, it is possible to transform the equation (3) to equation (1). Consequently, the triangles XS_1S_2 and XY_1Y_2 are similar (now by SAS theorem) and hence $S_1S_2 \parallel Y_1Y_2$. Therefore the distances of Y_1 and Y_2 to the line S_1S_2 are equal which proves the required property of the point X.

Answer. If $r_1 \neq r_2$, the locus of points X is the circle of Apollonius which is given given by the above equation (3), excepting the two points on the line S_1S_2. If $r_1 = r_2$, the locus is the perpendicular bisector of the segment S_1S_2, with exception of the midpoint of S_1S_2.

Remark. Let us prove directly the needed fact that any solution (i.e. point) X of the equation (3) lies in the exteriors of the circles k_1 and k_2. It follows easily from (3) that the differences $|XS_1| - r_1$ and $|XS_2| - r_2$ possess the same sign, which together with the following inequality

$$(|XS_1| - r_1) + (|XS_2| - r_2) \leqslant |S_1S_2| - (r_1 + r_2) > 0,$$

leads to the conclusion that both the differences are positive, i.e. $|XS_1| > r_1$ and $|XS_2| > r_2$, as promised to be proven.

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3. *Find all triples of real numbers x, y and z for which*

$$x(y^2 + 2z^2) = y(z^2 + 2x^2) = z(x^2 + 2y^2).$$

(Michal Rolínek)

Solution. If for example $x = 0$, we get a system $0 = yz^2 = 2y^2z$ which means that one of the unknowns y and z vanishes and the other can be arbitrary. The cases $y = 0$ or $z = 0$ are discussed similarly. Thus we have obtained three groups of solutions (x, y, z) which are formed by triples $(t, 0, 0)$, $(0, t, 0)$ and $(0, 0, t)$ respectively, where t is any real number. Moreover, we have observed that all the other solutions satisfy the condition $xyz \neq 0$, which is supposed to hold in what follows.

Factorizing the equation $x\bigl(y^2 + 2z^2\bigr) = y\bigl(z^2 + 2x^2\bigr)$ yields $(2x - y)(z^2 - xy) = 0$. Thus we distinguish two cases (depending on the fact which of the two factors vanishes).

(i) $2x - y = 0$. After setting $y = 2x$ the given system is reduced to the only equation

$$2x(2x^2 + z^2) = 9x^2 z,$$

which can be simplified (by dividing $x \neq 0$) to

$$4x^2 + 2z^2 - 9xz = 0 \quad \text{or} \quad (x - 2z)(4x - z) = 0.$$

Thus the case (i) yields exactly two groups of solutions $(2t, 4t, t)$ and $(t, 2t, 4t)$, where t is any real number.

(ii) $z^2 - xy = 0$. Substituting $z^2 = xy$ into the given system, we now get the only equation

$$xy(2x + y) = z(x^2 + 2y^2),$$

which is (because of the inequality $x^2 + 2y^2 > 0$) equivalent to

$$z = \frac{xy(2x + y)}{x^2 + 2y^2}.$$

At this moment we have to find when such a z obeys the condition $z^2 = xy$. After direct substitution we get the following condition on the unknowns x and y:

$$\frac{x^2y^2(2x + y)^2}{(x^2 + 2y^2)^2} = xy.$$

Dividing by $xy \neq 0$ and removing the fraction yields

$$xy(2x + y)^2 = (x^2 + 2y^2)^2 \quad \text{or} \quad (4y - x)(x^3 - y^3) = 0.$$

Thus we conclude that either $x = 4y$, or $x^3 = y^3$, i.e. $x = y$.[5] Returning to the formula for z, we obtain $z = 2y$ or $z = x$, according as $x = 4y$ or $x = y$. Consequently, there

[5] The reduction $x^3 = y^3$ to $x = y$ is correct, because the mapping $t \mapsto t^3$ is increasing on the set $\mathbb{R}$.

are two groups of solutions in the case (ii), namely triples $(4t, t, 2t)$ and (t, t, t), where t is any real number.

Answer. All the solutions are $(t, 0, 0)$, $(0, t, 0)$, $(0, 0, t)$, (t, t, t), $(4t, t, 2t)$, $(2t, 4t, t)$ a $(t, 2t, 4t)$, where t is any real number.

Remark. Let us describe a way how to avoid a more complicated case (ii) in the above solution. Thanks to the cyclic symmetry, the given system yields even three factorized equations

$$(2x - y)(z^2 - xy) = 0, \quad (2y - z)(x^2 - yz) = 0, \quad (2z - x)(y^2 - zx) = 0. \qquad (1)$$

The case $2x - y = 0$ were discussed as (i) in the above solution, the cases $2y - z = 0$ and $2z - x$ can be treated analogously. Summarizing the three cases, we get all the solutions indicated in *Answer*, excepting the triples (t, t, t). Thus for the remaining case when

$$z^2 - xy = x^2 - yz = y^2 - zx = 0 \qquad (2)$$

our task is to show that the only satisfactory triples are $(x, y, z) = (t, t, t)$. However, the last conclusion is an easy consequence of the dentity

$$(x - y)^2 + (y - z)^2 + (z - x)^2 = 2(z^2 - xy) + 2(x^2 - yz) + 2(y^2 - zx),$$

whose right-hand side vanishes by (2), and hence all the squares in the left-hand side vanish as well, which completes the proof. Let us add a note that the system (2) can be solved in another way: The equations in (2) easily imply that the values of x^3, y^3, z^3 are the same (namely, equal to the value of xyz), which happens only if $x = y = z$ by the footnote on previous page.

Another solution. To avoid unnecessary repetition from the above solution, we will solve the problem under the condition that $xyz \neq 0$.

Dividing both sides of the given equations by xyz we obtain

$$\frac{y}{z} + \frac{2z}{y} = \frac{z}{x} + \frac{2x}{z} = \frac{x}{y} + \frac{2y}{x}, \qquad (3)$$

which can be read as a coincidence of values of a function $f(s) = s + 2/s$ in three nonzero points $s_1 = y/z$, $s_2 = z/x$ a $s_3 = x/y$. Thus we first find when $f(s) = f(t)$ for two nonzero real numbers s and t. It follows from the identity

$$f(s) - f(t) = s + \frac{2}{s} - t - \frac{2}{t} = \frac{(s - t)(st - 2)}{st}$$

that $f(s) = f(t)$ if and only if $s = t$ or $st = 2$. Consequently, the system (3) holds if and only if the introduced numbers s_1, s_2, s_3 possess the following property: $s_i = s_j$ or $s_i s_j = 2$, for any indices i and j. However, if there exists a permutation (i, j, k) of $(1, 2, 3)$ such that $s_i s_j = 2$, then the identity $s_i s_j s_k = 1$ implies that $s_k = \frac{1}{2}$ and hence $s_i \in \{\frac{1}{2}, 4\}$ (because $s_i = s_k$ or $s_i s_k = 2$). Thus the assumption $s_i s_j = 2$ leads to the conclusion that (s_1, s_2, s_3) is a permutation of $(\frac{1}{2}, \frac{1}{2}, 4)$. It is easy to check that exactly three such permutations are satisfactory and yield the solutions $(4t, t, 2t)$, $(2t, 4t, t)$ and $(t, 2t, 4t)$ of the given system. In the remaining case when $s_1 = s_2 = s_3$, the identity $s_1 s_2 s_3 = 1$ implies that $s_i = 1$ for each i, which yields the solutions (t, t, t).

4. *Six teams will take part in a volleyball tournament. Each pair of the teams should play one match. All the matches will be realized in five rounds, each involving three simultaneous matches on the courts numbered 1, 2 and 3. Find the number of all possible draws for such a tournament. By a draw we mean a table* 5×3 *in which an unordered pair of teams is written on the field* (i, j), *where* $i \in \{1, 2, 3, 4, 5\}$ *and* $j \in \{1, 2, 3\}$, *if these two teams will meet each other in the* i*-th round on the court* j. *You are allowed to write down the resulting number as a product of prime factors (instead of writing its decimal expansion).*

(Martin Panák)

Solution. We postpone the question of permutations of the five rounds and the three courts to the end of our solution. Denoting first the teams by numbers 1, 2, 3, 4, 5, 6 (in a fixed way), we rearrange the five rounds of any satisfactory draw by means of the following numbering: Let 1 and 2 be the rounds with matches of the pairs of teams $(1, 2)$ and $(1, 3)$, respectively. If a pair $(3, a)$ plays in the round 1 and if a pair $(2, b)$ plays in the round 2, then a, b are two *distinct* numbers from $\{4, 5, 6\}$ (otherwise the third pairs in the rounds 1 and 2 are identical). Let 3, 4 and 5 denote the rounds with pairs $(1, a)$, $(1, b)$ and $(1, c)$ respectively, where $c \in \{4, 5, 6\} \setminus \{a, b\}$. Up to this moment, we have fixed an uncompleted draw

$$\begin{array}{lll} 1: & (1,2), & (3,a), \\ 2: & (1,3), & (2,b), \\ 3: & (1,a), & \\ 4: & (1,b), & \\ 5: & (1,c), & \end{array}$$

which can be extended to a the complete draw in the only one way:

$$\begin{array}{llll} 1: & (1,2), & (3,a), & (b,c), \\ 2: & (1,3), & (2,b), & (a,c), \\ 3: & (1,a), & (2,c), & (3,b), \\ 4: & (1,b), & (2,a), & (3,c), \\ 5: & (1,c), & (2,3), & (a,b). \end{array}$$

Since (a, b, c) is any permutation of $(4, 5, 6)$, the total number of the complete draws (written as above) is $3! = 6$. Taking in account the number $5!$ of the possible permutations of the five rounds and the number $3!$ of possible permutations of the three courts, we conclude that the requested number of all draws is equal to

$$6 \cdot 5! \cdot 6^5 = 5! \cdot 6^6 = 2^9 \cdot 3^7 \cdot 5 = 5\,598\,720.$$

Another solution. Let us denote the six teams by numbers 1, 2, 3, 4, 5 a 6 and construct first an "unordered" draw in which the rounds will be "numbered" by the opponents of the team 1 — see the following table in which the other opponents of the team 2 are denoted as a, b, c, d:

$$\begin{array}{lll} 1: & (1,2), & \\ 2: & (1,3), & (2,a), \\ 3: & (1,4), & (2,b), \\ 4: & (1,5), & (2,c), \\ 5: & (1,6), & (2,d). \end{array}$$

Note that (a, b, c, d) is a permutation of the quadruple $(3, 4, 5, 6)$ and that the following two restrictions are evident:

- ▷ $3 \neq a$, $4 \neq b$, $5 \neq c$ and $6 \neq d$
- ▷ The two-element sets $\{3, a\}$, $\{4, b\}$, $\{5, c\}$, $\{6, d\}$ are pairwise distinct.

It is clear that under these two conditions, the third pairs for the rounds 2–5 are uniquely determined, as well as the remaining two pairs for the round 1. Consequently, we have to calculate the number of permutations (a, b, c, d) of the quadruple $(3, 4, 5, 6)$ which satisfy the two above stated conditions.

Using the inclusion-exclusion principle we conclude that the first condition is fulfilled by exactly nine permutations:

$$4! - \left(4 \cdot 3! - \binom{4}{2} \cdot 2! + 4 - 1\right) = 9.$$

Moreover, exactly three of them do not satisfy the second condition, namely the permutations $(4, 3, 6, 5)$, $(5, 6, 3, 4)$ and $(6, 5, 4, 3)$. Thus the total number of the satisfactory permutations equals $9 - 3 = 6$.

We have proved that there are six "unordered" draws in the above specified sense. Combining this result with the idea of permuting the rounds and the courts, we conclude that the requested number of the draws is equal to

$$6 \cdot 6^5 \cdot 5! = 5! \cdot 6^6 = 2^9 \cdot 3^7 \cdot 5 = 5{,}598{,}720.$$

Remark. All the six satisfactory permutations (a, b, c, d) from the preceding solutions are $(4, 5, 6, 3)$, $(4, 6, 3, 5)$, $(5, 3, 6, 4)$, $(5, 6, 4, 3)$, $(6, 3, 4, 5)$, $(6, 5, 3, 4)$. It is possible to find them by an easy systematic examination (and thus to avoid the above presented calculation based on the inclusion-exclusion principle). On the other hand, the number 3 of the permutations (a, b, c, d) that satisfy the first, but not the second condition, can be determined as the number of the "faulty" equalities

$$\{3, a\} = \{4, b\}, \ \{3, a\} = \{5, c\}, \ \{3, a\} = \{6, d\},$$

which are successively equivalent to the others:

$$\{5, c\} = \{6, d\}, \ \{4, b\} = \{6, d\}, \ \{4, b\} = \{5, c\}.$$

Final Round of the 63rd Czech and Slovak Mathematical Olympiad (March 24–25, 2014)

1. *Let n be a natural number whose all positive divisors are denoted as $d_1, d_2, \dots, d_k$ in such a way that $d_1 < d_2 < \cdots < d_k$ (thus $d_1 = 1$ and $d_k = n$). Determine all the values of n for which both equalities $d_5 - d_3 = 50$ and $11d_5 + 8d_7 = 3n$ hold.*
(Matúš Harminc)

Solution. We distinguish whether n is odd or even.

(i) The case of n odd. Since all the d_i's are odd too, it follows from $11d_5+8d_7 = 3n$ that $d_7 \mid 11d_5$ as well as $d_5 \mid 8d_7$, hence $d_5 \mid d_7$. In view of $d_7 > d_5$, the relations $d_5 \mid d_7 \mid 11d_5$ imply that $d_7 = 11d_5$. Substituting this into $11d_5 + 8d_7 = 3n$, we obtain $99d_5 = 3n$ or $33d_5 = n$. Thus the four numbers 1, 3, 11 and 33 are divisors of n, more exactly all its divisors smaller that 50, since the fifth divisor d_5 satisfies $d_5 = d_3 + 50 > 50$. Consequently, it holds that $d_1 = 1$, $d_2 = 3$, $d_3 = 11$, $d_4 = 33$, $d_5 = d_3 + 50 = 61$, and thus $n = 33d_5 = 33 \cdot 61 = 2013$. The number 2013 is satisfactory indeed, because its first small divisors as indicated in the last sentence and moreover, the subsequent divisors are $d_6 = 61 \cdot 3$ and $d_7 = 61 \cdot 11$; hence $d_7 = 11d_5$ as required.

(ii) The case of n even. Now the equality $11d_5 + 8d_7 = 3n$ implies that $2 \mid d_5$ and hence $2 \mid d_5 - 50 = d_3$ as well. Since $d_1 = 1$, $d_2 = 2$ and $d_3 \neq 3$, we conclude that either $d_3 = 4$, or $d_3 = 2t$, with some integer $t > 2$. But the last is impossible (otherwise t is a divisor of n with $d_2 < t < d_3$, a contradiction), Therefore, we have $d_3 = 4$, $d_5 = d_3 + 50 = 54$ and hence 3 is a divisor of n between d_2 and d_3, a contradiction. In this way, the nonexistence of any satisfactory even n is proven.

Answer. The problem has the only solution $n = 2013$.

Another solution. The divisors d_5 and d_7 of n, with $d_5 < d_7$, can be represented as $d_5 = n/x$ and $d_7 = n/y$, where x and y ($x > y$) are some positive divisors of n again. Substituting this into $11d_5 + 8d_7 = 3n$, we obtain (after cancelling n) an equation $11/x + 8/y = 3$ which can be solved in a standard way, for example by a simple factorization:

$$8x = y(3x - 11) \;\Leftrightarrow\; 8(3x - 11) + 88 = 3y(3x - 11) \;\Leftrightarrow\; (3x - 11)(3y - 8) = 88.$$

The first equation implies that $3x - 11 > 0$ and hence $3y - 8 > 0$ as well. Note that $x \geqslant y + 1$ yields $3x - 11 \geqslant 3y - 8 > 0$. Taking in account the prime factorization $88 = 2^3 \cdot 11$, we conclude that the ordered pair $(3x-11, 3y-8)$ of factors must belong to the following set

$$\{(88, 1), (44, 2), (22, 4), (11, 8)\}.$$

However, congruences modulo 3 imply the only two pairs $(88, 1)$ and $(22, 4)$ are admissible. The corresponding pairs (x, y) are $(33, 3)$ and $(11, 4)$, respectively.

If $(x, y) = (33, 3)$, then $d_5 = n/33$ (and $d_7 = n/3$), thus 1, 3, 11 and 33 of divisors of n which leads (as in the above solution) to the solution $n = 2013$.

If $(x, y) = (11, 4)$, then $d_5 = n/11$ and $d_7 = n/4$, thus 1, 2, 4, 11, 22 and 44 are divisors of n, which contradicts to $d_5 > 50$.

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2. *We are given a segment AB in the plane. Consider a triangle XYZ with the following properties: the vertex X is an interior point of the segment AB, the triangles XBY and XZA are similar ($\triangle XBY \sim \triangle XZA$) and the points A, B, Y, Z lie on a circle in this order. Find the locus of midpoints of the sides YZ of all such triangles XYZ.* (Michal Rolínek, Jaroslav Švrček)

Solution. Let XYZ be a satisfactory triangle. Then the vertices Y and Z must lie in the same half-plane with the boundary line AB. Denote by Y' the reflection of Y through the line AB. Due to the presumed similarity, the angles XAZ a BYX are congruent (Fig. 1) and hence $|\angle BAZ| = |\angle BY'Z|$ as well. Using the well known inscribed angles property we conclude that the circumcircle k of $\triangle ABZ$ passes not only through the point Y, but also through the point Y'. The line AB (as a perpendicular bisector of the chord YY') passes through the centre O of the circle k and thus the chord AB is a diameter of k. Since the segment AB is fixed, the circle $k = ABYZ$ is common for all satisfactory triangles XYZ and the midpoint M of YZ must lie in the interior of k. Since the both angles OMZ and OMY are right (Fig. 2), the (lesser) angles AMO and BMO are acute and thus the point M must lie in the intersection of the exteriors of Thales' circles with diameters AO and BO. In what follows we will show the the both derived necessary conditions determine the locus of all the possible midpoints M.

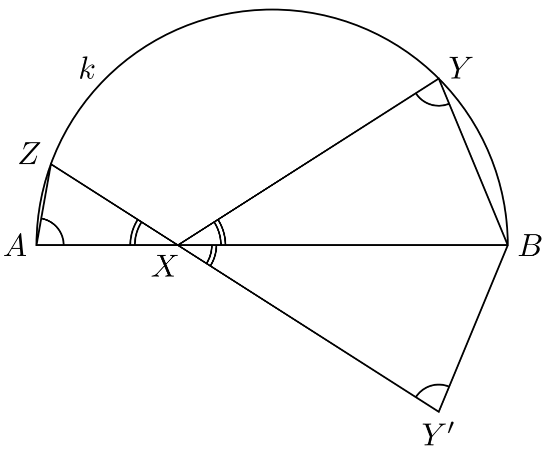

Fig. 1

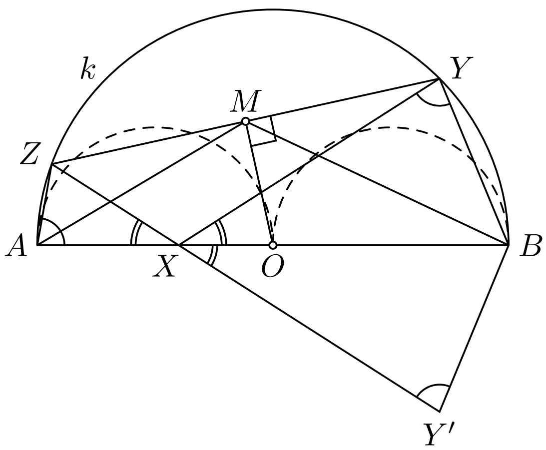

Fig. 2

So, let M be any point in the interior of k for which the both angles AMO and BMO are acute (i.e. M lies in the exteriors of the circles with diameters AO and BO). Consider a chord of k which passes through M perpendicularly to OM. This chord does not intersect the diameter AB, because of the acute angles AMO and BMO. Thus the endpoints of the chord with the midpoint M can be denoted as Y and Z so that A, B, Y, Z lie on k in this order. If Y reflects to Y' through the diameter AB and if X denotes the intersection point of the segments AB a $Y'Z$, then the triangles XBY a XZA are similar as required (by theorem AA). This completes the solution.

Conclusion. The locus under consideration is the interior of the highlighted region bounded by the three circles with diameters AB, AO and BO, where O denotes the midpoint of segment AB (Fig. 3).

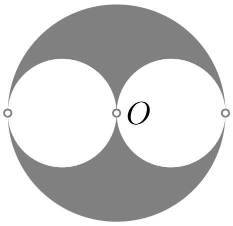

Fig. 3

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3. *Let us call by an "edge" any segment of length 1 which is common to two adjacent fields of a given chessboard 8×8. Consider all possible cuttings of the chessboard into 32 pieces 2×1 and denote by $n(e)$ the total number of such cuttings that involve the given edge e. Determine the last digit of the sum of the numbers $n(e)$ over all the edges e.* (Michal Rolínek)

Solution. The total number of the vertical edges is $7 \cdot 8 = 56$ as well as the total number of the horizontal edges. Thus the number of all the edges under consideration is $56 \cdot 2 = 112$.

The number of edges, which *are not* involved in a given cutting, is equal to 32, because each of these edges must coincide with the common segment of the two fields forming one of the 32 resulting pieces 2×1. Thus each cutting gives a contribution $112 - 32 = 80$ to the sum S of all the numbers $n(e)$. Consequently, the sum S is a multiple of 80 and thus its last digit is zero.

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4. *There are 234 visitors in a cinema auditorium. The visitors are sitting in n rows, where $n \geqslant 4$, so that each visitor in the i-th row has exactly j friends in the j-th row, for any $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. Find all the possible values of n. (Friendship is supposed to be a symmetric relation.)* (Tomáš Jurík)

Solution. For any $k \in \{1, 2, \ldots, n\}$ denote by p_k the number of visitors in the k-th row. The stated condition on given i and j implies that the number of friendly pairs (A, B), where A and B are from the i-th row and from j-th row respectively, is equal to the product jp_i. Interchanging the indices i and j, we conclude that the same number of friendly pairs (A, B) equals ip_j. Thus $jp_i = ip_j$ or $p_i : p_j = i : j$, and therefore, all the numbers p_k must be proportional as follows:

$$p_1 : p_2 : \cdots : p_n = 1 : 2 : \cdots : n.$$

Let us show that under this proportionality the visitors can be friendly in such a way which ensures the property under consideration. Thus assume that for some positive integer d, the equality $p_k = kd$ holds with any $k \in \{1, 2, \ldots, n\}$. Let us start with the case $d = 1$ when the numbers of visitors in single rows are successively $1, 2, \dots, n$. Then the stated property holds true if (and only if) any two visitors — taken from distinct rows in the whole auditorium — are friends. In the case when $d > 1$, let us divide all the visitors into d groups $G_1, G_2, \dots, G_d$ so that for arbitrary $k = 1, 2, \dots, d$, the numbers of visitors from the group G_k in single rows are successively $1, 2, \dots, n$. It is evident that the stated property holds true under the following condition: two visitors are friends if and only if they belong to the same group G_k.

It follows from the preceding that our task is to find such integer values of n, $n \geqslant 4$, for which there exists a positive integer d satisfying the equation

$$d + 2d + \cdots + nd = 234 \quad \text{or} \quad dn(n+1) = 468.$$

Thus we look for all divisors $468 = 2^2 \cdot 3^2 \cdot 13$ which are of the form $n(n+1)$. Inequality $22 \cdot 23 > 468$ implies that $n < 22$ and hence $n \in \{4, 6, 9, 12, 13, 18\}$. It is easy to see that the only satisfactory n equals 12 (which corresponds to $d = 3$).

Answer. The unique solution is $n = 12$.

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5. *We are given an acute-angled triangle ABC. Denote by k the circle with diameter AB. A circle touching the bisector of the angle BAC at the point A and passing through the point C meets the circle k in a point P, $P \neq A$. Similarly, a circle touching the bisector of the angle ABC at the point B and passing through the point C meets the circle k in a point Q, $Q \neq B$. Prove that the lines AQ and BP intersect each other on the bisector of the angle ACB.* (Peter Novotný)

Solution. Besides the Thales' circle k, denote as $l_A = APC$ and $l_B = BQC$ the other two circles under consideration. Let us deal, for example, with the circle l_B drawn in Fig. 4. In what follows the interior angles of ΔABC are denoted as usual.

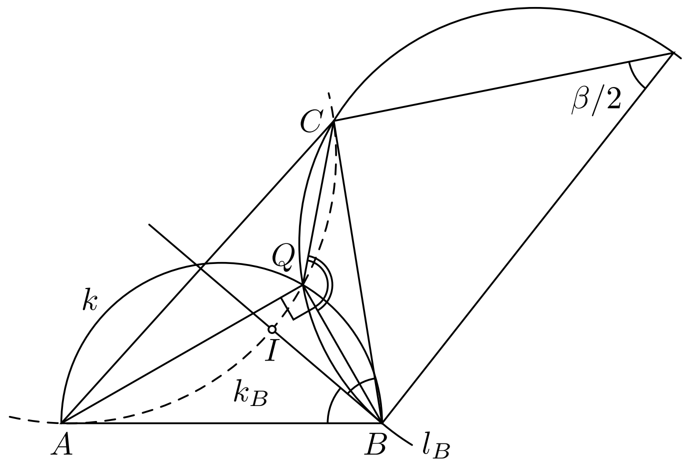

Fig. 4

Let us explain that it is true indeed what the figure suggests. First of all, the point Q lies in the half-plane BCA, because the local arc BC of the circle l_B has the following property: as its point X moves from C to B, the angle AXB varies from an acute angle γ to an obtuse angle $180° - \beta/2$, and hence the Thales' circle k meets the arc BC in an interior point Q. Applying the properties of inscribed and subtended angles to the chord BC of the circle l_B, we conclude that $|\angle BQC| = 180^\circ - \beta/2$ and hence $|\angle AQB| + |\angle BQC| = 270^\circ - \beta/2 > 180^\circ$. Consequently, Q is a point of the half-plane ACB which lies in the interior of the triangle ABC and the convex angle AQC equals $90^\circ + \beta/2$. As known, the last is a measure of the angle AIC, where I denotes the incentre of ΔABC (indeed, $|\angle AIC| = 180^\circ - \alpha/2 - \gamma/2 = 90^\circ + \beta/2$). In this way, the congruence of the angles AQC and AIC is proven and hence the points Q and I lie on the same arc AC of a new circle $k_B = ACI$. Therefore, the line AQ is the radical axis of the circles k and k_B. Analogously, the line BP is the radical axis of the circles k and $k_A = BCI$.

It remains to note that the intersection point of the lines AQ and BP (treated as the above radical axes) has the same power with respect to the circles k_A and k_B, whose radical axis is the line CI, i.e. just the bisector of the angle ACB. This completes our solution.

Remark. Let us explain once again that the point Q lies in the open half-plane BCA. The intersection point Q of the circles k and l_B is clearly an interior point of the half-plane ABC which lies between the two lines tangent to k and l_B at the point B. Note that both the vertex C of the acute-angled triangle ABC and the centre S_B of the circle l_B lie in the exterior of the Thales' circle k. An arc of k lies in the triangle BS_BC which however does not contain any point of the circle l_B, excepting the points B and C. Consequently, the point Q has to lie in the half-plane BCA.

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6. *Let a, b be non-negative real numbers. Prove the inequality*

$$\frac{a}{\sqrt{b^2+1}} + \frac{b}{\sqrt{a^2+1}} \geqslant \frac{a+b}{\sqrt{ab+1}}$$

and find when the equality holds. (Tomáš Jurík, Jaromír Šimša)

Solution. It is evident that the inequality under consideration becomes an equality when $a = 0$, $b = 0$ or $a = b$. To prove that otherwise the strong inequality holds, it suffices to deal with the case $a > b > 0$ and (after removing the fractions) to show that

$$a\sqrt{a^2+1}\sqrt{ab+1} + b\sqrt{b^2+1}\sqrt{ab+1} > (a+b)\sqrt{a^2+1}\sqrt{b^2+1}.$$

Distributing the right-hand side and regrouping the terms we get

$$a\sqrt{a^2+1}\big(\sqrt{ab+1} - \sqrt{b^2+1}\big) > b\sqrt{b^2+1}\big(\sqrt{a^2+1} - \sqrt{ab+1}\big).$$

Multiplying the differences of the square roots by their sums as the denominators of new installed fractions, we obtain

$$a\sqrt{a^2+1} \cdot \frac{b(a-b)}{\sqrt{ab+1} + \sqrt{b^2+1}} > b\sqrt{b^2+1} \cdot \frac{a(a-b)}{\sqrt{a^2+1} + \sqrt{ab+1}}.$$

Dividing both sides by the positive number $ab(a-b)$ and removing the fractions again, we finally arrive at an equivalent inequality

$$\sqrt{a^2+1}\big(\sqrt{a^2+1} + \sqrt{ab+1}\big) > \sqrt{b^2+1}\big(\sqrt{b^2+1} + \sqrt{ab+1}\big),$$

which easily follows by an easy comparison of the both sides "term by term" (because our assumption $a > b$ implies that $\sqrt{a^2+1} > \sqrt{b^2+1}$). This completes the proof of the given inequality. As we have shown, the only cases of the equality are $a = 0$, $b = 0$ and $a = b$.

Another solution. We exclude the cases $a = 0$ and $b = 0$ (when the inequality becomes an equality) from our considerations. Let us apply a Cauchy-Schwarz inequality in the form

$$\Big(\frac{a}{u} + \frac{b}{v}\Big)(au + bv) \geqslant (a+b)^2,$$

with positive coefficients $u = \sqrt{b^2+1}$ and $v = \sqrt{a^2+1}$:

$$\left(\frac{a}{\sqrt{b^2+1}} + \frac{b}{\sqrt{a^2+1}}\right)\left(a\sqrt{b^2+1} + b\sqrt{a^2+1}\right) \geqslant (a+b)^2. \tag{1}$$

Another Cauchy-Schwarz inequality yields an upper bound for the second factor from the-left hand side of (1):

$$a\sqrt{b^2+1} + b\sqrt{a^2+1} = \sqrt{a}\sqrt{ab^2+a} + \sqrt{b}\sqrt{a^2b+b} \leqslant$$
$$\leqslant \sqrt{a+b}\sqrt{ab^2+a+a^2b+b} = \sqrt{a+b}\sqrt{(a+b)(ab+1)} = (a+b)\sqrt{ab+1}.$$

Consequently, the first factor in (1) has a lower bound

$$\frac{a}{\sqrt{b^2+1}} + \frac{b}{\sqrt{a^2+1}} \geqslant \frac{(a+b)^2}{a\sqrt{b^2+1} + b\sqrt{a^2+1}} \geqslant \frac{a+b}{\sqrt{ab+1}},$$

which is the desired inequality. Since (1) becomes an equality if and only if the positive coefficients u and v are the same, i.e. $\sqrt{b^2+1} = \sqrt{a^2+1}$ in our situation, the equality $a = b$ is the third (and last) case (next to $a = 0$ and $b = 0$ from the introductory sentence) when the proven inequality holds as an equality.

Another solution. Let us exclude the obvious cases $a = 0$, $b = 0$, $a = b$ and let us transform the (strong) inequality under consideration into the following equivalent form:

$$\frac{a}{a+b} \cdot \frac{1}{\sqrt{b^2+1}} + \frac{b}{a+b} \cdot \frac{1}{\sqrt{a^2+1}} > \frac{1}{\sqrt{ab+1}}. \tag{2}$$

The last left-hand side can be read as that of the (strong) Jensen inequality

$$pf(\alpha) + qf(\beta) > f(p\alpha + q\beta), \tag{3}$$

with positive coefficients $p = a/(a+b)$ and $q = b/(a+b)$ (which satisfy $p + q = 1$ as required), applied to the function $f(x) = 1/\sqrt{x}$ at the points $\alpha = b^2+1$ and $\beta = a^2+1$. Since the function f is strictly convex[6] on the interval $(0, +\infty)$ and since the points α and β are assumed to be distinct, the Jensen inequality (3) holds.

It remains to verify that also the right-hand sides of (2) and (3) are identical. This is easy:

$$f(p\alpha + q\beta) = f\Big(\frac{a}{a+b}(b^2+1) + \frac{b}{a+b}(a^2+1)\Big) =$$
$$= f\left(\frac{a + ab^2 + b + a^2b}{a+b}\right) = f(ab+1) = \frac{1}{\sqrt{ab+1}}.$$

[6] The shape of the curve $y = x^{-\frac{1}{2}}$ is well known from high-school textbooks.