2015

64th Czech and Slovak Mathematical Olympiad

First Round of the 64th Czech and Slovak Mathematical Olympiad
Problems for the take-home part

1. A natural number n is given. Square with side of length n is divided into n^2 unit squares. For the distance between two squares we consider the distance from centre to centre. Find the number of pairs of squares whose distance is 5.
(Jaroslav Zhouf)

2. A triangle ABC is given in which BC is the shortest side. Denote M its midpoint. On the sides AB and AC take the points X and Y, respectively, in such a way that $|BX| = |BC| = |CY|$. Denote Z the intersection point of lines CX and BY. Prove that the line ZM passes through the centre of the excircle escribed to the side BC of the triangle. *(Michal Rolínek)*

3. Find all integers $k \geqslant 2$ for which there exists k-element set M of positive integers such that the product of all numbers in M is divisible by the sum of any two (different) numbers from M. *(Jaromír Šimša)*

4. Suppose that the real numbers x, y, z satisfy equalities

$$15(x + y + z) = 12(xy + yz + zx) = 10(x^2 + y^2 + z^2)$$

and that at least one of them is different from zero.

a) Prove that $x + y + z = 4$.

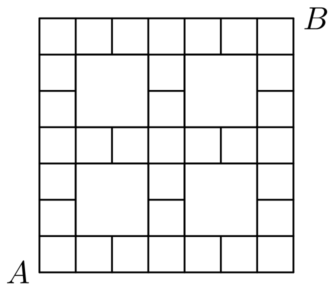
b) Find the smallest interval $\langle a, b\rangle$, which contains all three numbers from any triplet (x, y, z) satisfying the given conditions. *(Jaromír Šimša)*

5. In the triangle ABC denote D point of contact of side BC with the incircle. The incircle of the triangle ABD is tangent to sides AB and BD at points K and L. The incircle of the triangle ADC is tangent to sides DC and AC at points M and N. Prove that points K, L, M, N lie on the same circle. *(Josef Tkadlec)*

6. Let a, b be relatively prime integers. Sequence $(x_n)_{n=1}^{\infty}$ of natural numbers is constructed in such a way that for each $n > 1$ applies $x_n = ax_{n-1} + b$. Prove that in any such sequence every entry x_n with index $n > 1$ divides infinitely many of other entries. Does this assertion hold for $n = 1$? *(Jaromír Šimša)*

First Round of the 64th Czech and Slovak Mathematical Olympiad (December 9th, 2014)

1. Find the number of paths of length 14 that run along the edges of the network in the picture from point A to point B. The length of each edge is one.



(Pavel Novotný)

2. A parallelogram $ABCD$ with $|AB| = 2|BC|$ is given. Determine all the lines that divide the parallelogram into two tangential quadrilaterals. *(Jaroslav Švrček)*

3. Determine all pairs (p, q) of integers such that p is an integer multiple of q and quadratic equation $x^2 + px + q = 0$ has at least one integer root.

(Jaroslav Švrček)

Second Round of the 64th Czech and Slovak Mathematical Olympiad (January 22nd, 2015)

1. A triangle ABC with obtuse angle at C is given. Axis o_1 of side AC intersects side AB in point K, axis o_2 of side BC intersects side AB in point L. Denote O intersection of the axes o_1 and o_2. Prove that centre of the incircle of triangle KLC lies on the circumcircle of triangle OKL. (*Radek Horenský*)

2. Find all the pairs of prime numbers (p, q) such that the value of the expression $p^2 + 4q + 5pq^2$ is squared integer. (*Pavel Calábek*)

3. For positive real numbers a, b, c the following holds:

$$ab + bc + ca = 16, \quad a \geqslant 3.$$

Find the smallest possible value of the expression $2a + b + c$. (*Michal Rolínek*)

4. We are given n points in a plane, $n \geqslant 3$, no three of them collinear. Consider all the interior angles of all triangles with vertices in given points and denote ϕ the size of the smallest angle. For given n find the largest possible ϕ.
(*Stanislava Sojáková*)

Final Round of the 64th Czech and Slovak Mathematical Olympiad (March 23–24, 2015)

1. Find all four-digit numbers n satisfying the following conditions:

i) number n is product of three different primes;

ii) sum of the two smallest of these prime numbers is equal to the difference of largest two of them;

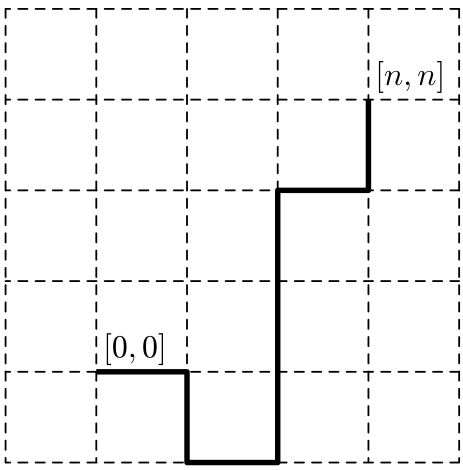
iii) sum of three primes is equal to the square of another prime.

(*Radek Horenský*)

2. For a given natural number n specify the number of paths of length $2n + 2$ from point $[0, 0]$ to the point $[n, n]$ which do not pass any point more than once. Path of length $2n + 2$ connecting points $[0, 0]$ and $[n, n]$ means $(2n + 2)$-tuple

$$(A_0A_1, A_1A_2, A_2A_3, \cdots, A_{2n+1}A_{2n+2})$$

of line segments connecting two adjacent lattice points, while $A_0 = [0, 0]$, $A_{2n+2} = [n, n]$. (*Pavel Novotný*)



3. In any triangle ABC, in which the median from C is not perpendicular to any of the sides CA nor CB, let us denote X and Y intersections of this median's axis with lines CA and CB. Find all such triangles ABC for which points A, B, X, Y lie on the same circle. (*Ján Mazák*)

4. In the field of real numbers solve a system of equations

$$\begin{aligned} a(b^2+c) &= c(c+ab),\\ b(c^2+a) &= a(a+bc),\\ c(a^2+b) &= b(b+ca). \end{aligned}$$

(*Michal Rolínek*)

5. A triangle ABC is given every two sides of which differ in length by at least $d > 0$. Denote by T its centroid, I incentre and ρ inradius. Prove that

$$S_{AIT} + S_{BIT} + S_{CIT} \geqslant \frac{2}{3}\rho d,$$

where S_{XYZ} denotes the area of triangle XYZ. (*Michal Rolínek*)

6. We are given a positive integer $n > 2$. Find the greatest of all the numbers d, satisfying the following condition: For any set of n integers one can choose its three different subsets so that the sum of elements each of which is an integer multiple of d. (The selected subsets need not be disjoint.) (*Jaromír Šimša*)