Czech and Slovak

2021 / 2022

First Round (take-home part), October 2021

1. Is it possible to fill an $n \times n$ table with numbers 1 and 2 such that the sum of the numbers in each column is a multiple of 5 and the sum of the numbers in each row is a multiple of 7? Solve this:
a) for $n = 9$, b) for $n = 12$. (*Tomáš Bárta*)

2. Let $ABCD$ be a trapezoid ($AB \parallel CD$) and denote $P = BC \cap AD$. Let k_1, k_2 be circles with diameters BC, AD, respectively, and denote $P = BC \cap AD$. Prove that the tangents from P to k_1 form the same angle as the tangents from P to k_2. (*Patrik Bak*)

3. Find all integers $n > 2$ such that n^{n-2} is an nth power of some integer. (*Patrik Bak*)

4. Find all triples (x, y, z) of real numbers such that

$$xy + 1 = z^2,$$
$$yz + 2 = x^2,$$
$$zx + 3 = y^2.$$

(*Tomáš Jurík*)

5. Let ABC be a scalene triangle, I its incenter and k its circumcircle. Rays BI, CI meet k again at $S_b \neq B$, $S_c \neq C$, respectively. Prove that the tangent to k at A, the line S_bS_c, and the line through I parallel to BC are concurrent. (*Patrik Bak*)

6. Consider an infinite sequence $a_0, a_1, a_2 \dots$ of integers that satisfies $a_0 \geq 2$ and $a_{n+1} \in \{2a_n - 1, 2a_n + 1\}$ for all indices $n \geq 0$. Prove that any such infinite sequence contains infinitely many composite numbers. (*Martin Melicher, Josef Tkadlec*)

First Round (on-site part), December 7th, 2021

1. Find the largest integer d for which a 43×47 table can be filled with numbers 1 and 2 such that the sum of the numbers in each column and in each row is a multiple of d. (Do not forget to show that no larger d works.) (*Tomáš Bárta*)

2. Let ABC be an acute triangle and I its incenter. Rays BI, CI meet the circumcircle of triangle ABC again at $S \neq B$, $T \neq C$, respectively. The segment ST meets the sides AB, AC at K, L, respectively. Prove that $AKIL$ is a rhombus. (*Josef Tkadlec*)

3. Find all pairs of positive integers a, b such that $a^{a-b} = b^a$. (*Jaromír Šimša*)

Second Round, January 11th, 2022

1. Is it possible to fill the 8×8 table with numbers 6 and 7 such that the sum of the numbers in each column is a multiple of 5 and the sum of the numbers in each row is a multiple of 7? (*Josef Tkadlec*)

2. Find all triples (x, y, z) of positive real numbers such that

$$\begin{aligned} x^2 + 2y^2 &= x + 2y + 3z, \\ y^2 + 2z^2 &= 2x + 3y + 4z, \\ z^2 + 2x^2 &= 3x + 4y + 5z. \end{aligned}$$

(*Patrik Bak*)

3. Let ABC be an isosceles triangle with base AB and P a point on its C-altitude. Ray AP meets the circumcircle of the triangle ABC again at $Q \neq A$. The line through P parallel to AB meets the side BC at R. Prove that QR bisects the angle AQB. (*Jaroslav Švrček*)

4. Consider an infinite sequence $a_0, a_1, a_2 \ldots$ of integers that satisfies

$$a_0 \geq 1 \quad \text{and} \quad a_{n+1} \in \{2022a_n - 1, 2022a_n + 1\}$$

for all indices $n \geq 0$. Prove that any such infinite sequence contains infinitely many composite numbers. (*Martin Melicher*)

Final Round, March 20th–23rd, 2022

1. In a sequence of 71 nonzero real numbers, each number (apart from the first one and the last one) is one less than the product of its two neighbors. Prove that the first and the last number are equal. (*Josef Tkadlec*)

2. We say that a positive integer k is *fair* if the number of 2021digit palindromes that are a multiple of k is the same as the number of 2022digit palindromes that are a multiple of k. Does the set $M = \{1, 2, \ldots, 35\}$ contain more numbers that are fair or those that are not fair?
(A palindrome is an integer that reads the same forward and backward.) (*David Hruška*)

3. Given a scalene acute triangle ABC, let M be the midpoints of its side BC and N the midpoint of the arc BAC of its circumcircle. Let ω be the circle with diameter BC and D, E its intersections with the bisetor of angle BAC. Points D', E' lie on ω such that $DED'E'$ is a rectangle. Prove that D', E", M, N lie on a single circle (*Patrik Bak*)

4. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$ and P its intersection of diagonals. Denote by O_1, O_2 the circumcenters of triangles ABP, CDP, respectively. Prove that O_1BCO_2 is a parallelogram. (*Patrik Bak*)

5. Find all integers n such that $2^n + n^2$ is a square of an integer. (*Tomáš Jurík*)

6. Consider any graph with 50 vertices and 225 edges. We say that a triplet of its (mutually distinct) vertices is *connected* if the three vertices determine at least two edges. Determine the smallest and the largest possible number of connected triples. (*Ján Mazák, Josef Tkadlec*)

Czech-Polish-Slovak Match, July 1st–4th, 2022

Guest countries: Austria, Ukraine

Institute of Science and Technology Austria (ISTA)

1. Let $k \leq 2022$ be a positive integer. Alice and Bob play a game on a 2022×2022 board. Initially, all cells are white. Alice starts and the players alternate. In her turn, Alice can either color one white cell in red or pass her turn. In his turn, Bob can either color a $k \times k$ square of white cells in blue or pass his turn. Once both players pass, the game ends and the person who colored more cells wins (a draw can occur). For each $1 \leq k \leq 2022$, determine which player (if any) has a winning strategy. (*David Hruška*)

2. Find all functions $f\colon (0,\infty) \to (0,\infty)$ such that

$$f\left(f(x) + \frac{y+1}{f(y)}\right) = \frac{1}{f(y)} + x + 1$$

for all $x, y > 0$. (*Dominik Burek*)

3. Circles Ω_1 and Ω_2 with different radii intersect at two points, denote one of them by P. A variable line ℓ passing through P intersects the arc of Ω_1 which is outside of Ω_2 at X_1, and the arc of Ω_2 which is outside of Ω_1 at X_2. Let R be the point on segment X_1X_2 such that $X_1P = RX_2$. The tangent to Ω_1 through X_1 meets the tangent to Ω_2 through X_2 at T. Prove that line RT is tangent to a fixed circle, independent of the choice of ℓ. (*Josef Tkadlec*)

4. Given a positive integer n, denote by $\tau(n)$ the number of positive divisors of n, and by $\sigma(n)$ the sum of all positive divisors of n. Find all positive integers n satisfying

$$\sigma(n) = \tau(n) \cdot \left\lceil \sqrt{n} \right\rceil.$$

(Here, $\lceil x \rceil$ denotes the smallest integer not less than x.) (*Michael Reitmeir*)

5. Let ABC be a triangle with $AB < AC$ and circumcenter O. The angle bisector of $\angle BAC$ meets the side BC at D. The line through D perpendicular to BC meets the segment AO at X. Furthermore, let Y be the midpoint of segment AD. Prove that points B, C, X, Y lie on a single circle. (*Karl Czakler*)

6. Consider 26 letters $A, \ldots, Z$. A *string* is a finite sequence consisting of those letters. We say that a string s is *nice* if it contains each of the 26 letters at least once, and each permutation of letters $A, \ldots, Z$ occurs in s as a subsequence the same number of times. Prove that:

(a) There exists a nice string.

(b) Any nice string contains at least 2022 letters.

(Here, a permutation π of the 26 letters is a *subsequence* of a string $s = s_1 s_2 s_3 \ldots$ if there exist 26 indices $i_1 < i_2 < \cdots < i_{26}$ such that $\pi = s_{i_1} s_{i_2} \ldots s_{i_{26}}$.) (*Václav Rozhoň*)

Statistics.

Second Round: 193 contestants, maximum score $(6, 6, 6, 6)$, mean scores $(5.26, 4.42, 1.95, 0.87)$.

Final Round: 49 contestants, maximum score $(7, 7, 7, 7, 7, 7)$, mean scores $(6.04, 4.78, 1.78, 4.82, 3.33, 0.86)$.

CPS Match: 30 contestants, maximum score $(7, 7, 7, 7, 7, 7)$, mean scores $(6.33, 2.17, 0.93, 6.13, 4.27, 0.60)$.