Czech-Polish-Slovak Match 2016
Day 1

1. Let P be a non-degenerate polygon with n sides, where $n > 4$. Prove that there exist three distinct vertices A, B, C of P with the following property: If ℓ_1, ℓ_2, ℓ_3 are the lengths of the three polygonal chains into which A, B, C break the perimeter of P, then there is a triangle with side lengths ℓ_1, ℓ_2, and ℓ_3.

   Remark: By a non-degenerate polygon we mean a polygon in which every two sides are disjoint, apart from consecutive ones, which share only the common endpoint.

   (Poland)

2. Let $m, n > 2$ be even integers. Consider a board of size $m \times n$ whose every cell is colored either black or white. The Guesser does not see the coloring of the board but may ask the Oracle some questions about it. In particular, she may inquire about two adjacent cells (sharing an edge) and the Oracle discloses whether the two adjacent cells have the same color or not. The Guesser eventually wants to find the parity of the number of adjacent cell-pairs whose colors are different. What is the minimum number of inquiries the Guesser needs to make so that she is guaranteed to find her answer?

   (Czech Republic)

3. Let n be a positive integer. For a finite set M of positive integers and each $i \in \{0, 1, \ldots, n-1\}$, we denote s_i the number of non-empty subsets of M whose sum of elements gives remainder i after division by n. We say that M is n-balanced if $s_0 = s_1 = \cdots = s_{n-1}$. Prove that for every odd number n there exists a non-empty n-balanced subset of $\{1, 2, \ldots, n\}$.

   For example if $n = 5$ and $M = \{1, 3, 4\}$, we have $s_0 = s_1 = s_2 = 1$, $s_3 = s_4 = 2$ so M is not 5-balanced.

   (Czech Republic)

Czech-Polish-Slovak Match 2016
Day 2

1. Find all quadruplets (a, b, c, d) of real numbers satisfying the system

$$(a+b)(a^2+b^2) = (c+d)(c^2+d^2),$$
$$(a+c)(a^2+c^2) = (b+d)(b^2+d^2),$$
$$(a+d)(a^2+d^2) = (b+c)(b^2+c^2).$$

(Slovakia)

2. Prove that for every non-negative integer n there exist integers x, y, z with $gcd(x, y, z) = 1$, such that

$$x^2 + y^2 + z^2 = 3^{2^n}.$$

(Poland)

3. Let ABC be an acute-angled triangle with $AB < AC$. Tangent to its circumcircle Ω at A intersects the line BC at D. Let G be the centroid of ABC and let AG meet Ω again at $H \neq A$. Suppose the line DG intersects the lines AB and AC at E and F, respectively. Prove that $\angle EHG = \angle GHF$.

(Slovakia)

Solutions

1. By scaling, we can assume w.l.o.g. that the perimeter of P has length 2. Let $X_1, \ldots, X_n$ be the vertices of P, and let $x_i = |X_i X_{i+1}|$, where $X_{n+1} = X_1$; then $\sum_{i=1}^n x_i = 2$. Since P is a non-degenerate polygon, we have that $x_i < 1$ for all $i = 1, 2, \ldots, n$. To prove the claim it is sufficient to partition the cyclic sequence $(x_1, \ldots, x_n)$ into three intervals such that the sum in each interval is strictly smaller than 1; the endpoints of these intervals correspond to the selected vertices A, B, C of P. To this end, we distinguish two cases: either there is some p such that $\sum_{i=1}^p x_i = 1$, or there is no such p.

   In the first case, the perimeter of P can be broken into two polygonal chains of length 1 each, both with endpoints X_1 and X_{p+1}. Since $x_i < 1$ for all i, both these chains consist of at least two segments. Since $n > 4$, we infer that the four segments incident to X_1 and X_{p+1}, namely $X_n X_1$, $X_1 X_2$, $X_p X_{p+1}$ and $X_{p+1} X_{p+2}$, are pairwise different, and they do not constitute the whole perimeter. Hence $x_n + x_1 + x_p + x_{p+1} < 2$, so either $x_n + x_1 < 1$ or $x_p + x_{p+1} < 1$. In the former subcase we can take intervals (x_n, x_1), $(x_2, \ldots, x_p)$ and $(x_{p+1}, \ldots, x_{n-1})$, and in each of them the sum is clearly smaller than 1. Symmetrically, in the latter subcase we can take intervals (x_p, x_{p+1}), $(x_{p+2}, \ldots, x_n)$, and $(x_1, \ldots, x_{p-1})$.

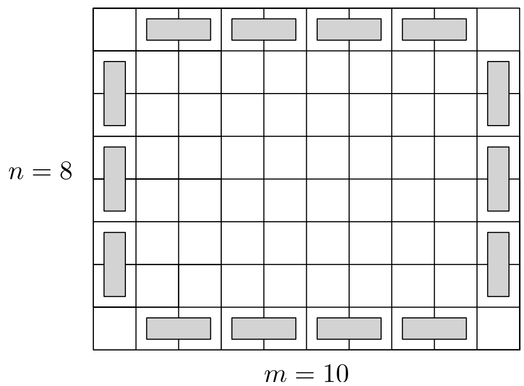
   We are left with the second case. Let q be the largest index such that $\sum_{i=1}^q x_i < 1$. Since $x_1 < 1$ and $x_n < 1$, we have that $1 \leq q \leq n-2$. By the choice of q and the fact that the first case was not applicable, we have that $\sum_{i=1}^{q+1} x_i > 1$, hence $\sum_{i=q+2}^n x_i = 2 - \sum_{i=1}^{q+1} x_i < 1$. As $x_{q+1} < 1$, we can take intervals $(x_1, \ldots, x_q)$, (x_{q+1}), and $(x_{q+2}, \ldots, x_n)$, and in each of them the sum is strictly smaller than 1. $\square$

2. Let b be the number of adjacent cell pairs with different color. We begin with the following observation. If we change a color of a cell with an even number of neighbours, then an even number of cell pairs change their state, implying that b mod 2 remains the same. The same argument applies if we flip the color of every cell inside a region that has an even number of neighbouring cells.

   Let us denote by E and O the sets of cells with even and odd number of neighbours, respectively. Note that O consists of the non-corner perimeter cells and let us partition it into pairs $O_1, O_2, \ldots, O_{m+n-4}$ as hinted by the picture. Then the following claim is crucial.

   Claim. Let b' be the number of differently colored pairs amongst $O_1, \ldots, O_{m+n-4}$. Then $b' \equiv b \pmod{2}$.

   Proof. We apply a sequence of flips that do not change b mod 2. First, let us flip (one by one) each of the white cells in E to black. Then for any adjacent pair (not necessarily one of $O_1, \ldots, O_{m+n-4}$) of white cells in O, flip them both to black. Now white cells are only in O and each of them has three black neighbours. But then both b and b' have clearly the same parity as the (current) number of white cells! $\square$



The rest is straightforward. Trivially, we can find b' in $m+n-4$ questions by asking about each of the pairs $O_1, \ldots, O_{m+n-4}$.

For the converse argument assume that fewer inquiries can determine the answer. Then some cell $c \in O$ was not involved in any inquiry. If we now consider two possible colorings of the board, the true one (according to which the Oracle is answering) and one that differs from it only at c, we see that these colorings have a different value of $b' \bmod 2$ (thus also of $b \bmod 2$) but cannot be distinguished based on the answers provided by the Oracle.

This concludes the solution. The minimum number of inquiries is $m+n-4$.

□

3. For M to be n-balanced, the number of its non-empty subsets must be a multiple of n, so $s = |M|$ must satisfy $n \mid 2^s - 1$. Since the sequence $2^m \pmod{n}$ is for odd n (in both directions) periodic and $2^0 = 1$, such $0 < s \leq n$ exists.

   Take smallest such $0 < s \leq n$, let us set
   $$M' = \{2^0, 2^1, \ldots, 2^{s-1}\}.$$
   Since every positive integer has a unique binary representation, the sums of subsets of M' are exactly the numbers $1, 2, 3, \ldots, 2^s - 1$, each appearing exactly once. Since by construction $n \mid 2^s - 1$, we can construct M by replacing each $m' \in M'$ by its residue mod n. By minimality of s, these residues are pairwise distinct and also such M is n-balanced. We are done. □

4. Let us set
   $$f(x,y) = (x+y)(x^2+y^2).$$
   We'll show that for any real x the inequality $y \geq z$ implies $f(x,y) \geq f(x,z)$. After subtraction we see that
   $$f(x,y) - f(x,z) = \frac{1}{2}(y-z)\left((x+y)^2 + (y+z)^2 + (z+x)^2\right) \geq 0.$$
   Moreover, equality occurs when $y = z$ or $x = y = z = 0$, so either way it implies $y = z$.

We can rewrite the system (implicitly using the symmetry of f) to the form:

$$f(a,b) = f(c,d)$$
$$f(a,c) = f(b,d)$$
$$f(a,d) = f(b,c)$$

Now we can see that the system is symmetric in variables a, b, c, d and may assume $a = \max\{a,b,c,d\}$. We then write the chain of (in)equalities

$$f(c,d) = f(a,b) \geq f(c,b) = f(a,d) \geq f(b,d) = f(a,c) \geq f(d,c)$$

and since we in fact have equality everywhere, we deduce $a = b = c = d$.

All such quadruplets clearly satisfy the system so the problem is solved. □

5. We will use the following algebraic formula:

$$(x^2 + y^2 + z^2)^2 = (x^2 + y^2 - z^2)^2 + (2xz)^2 + (2yz)^2.$$

This means that if a positive integer can be represented as a sum of 3 squares, then so can its square. Consequently, if we put

$$\begin{aligned}(x_0, y_0, z_0) &= (1,1,1)\\ (x_{n+1}, y_{n+1}, z_{n+1}) &= (x_n^2 + y_n^2 - z_n^2, 2x_nz_n, 2y_nz_n),\end{aligned}$$

then by a trivial induction we obtain that $x_n^2 + y_n^2 + z_n^2 = 3^{2^n}$.

It remains to show that $\gcd(x_n, y_n, z_n) = 1$ for all n and we proceed by induction.

Suppose $\gcd(x_n, y_n, z_n) = 1$, but x_{n+1}, y_{n+1}, and z_{n+1} have some common prime divisor p. Observe that since 3^{2^n} is odd, but $2x_nz_n$ and $2y_nz_n$ are even, we have that $x_n^2 + y_n^2 - z_n^2$ is odd, so $p \neq 2$. Hence $p|x_nz_n$ and $p|y_nz_n$. Then either $p|z_n$, or $p|x_n$ and $p|y_n$. In the latter case we could infer from $p|x_n^2 + y_n^2 - z_n^2$ that in fact also $p|z_n$, which contradicts the assumption that $\gcd(x_n, y_n, z_n) = 1$. Hence we are left with the first case: $p|z_n$.

Since $p|x_n^2 + y_n^2 - z_n^2$ and $p|z_n$, we also have that $p|x_n^2 + y_n^2 + z_n^2 = 3^{2^n}$. Hence in fact $p = 3$. But the only quadratic residues modulo 3 are 0 and 1, so the two possibilities for a sum of three squares to be divisible by 3 is that either all or none of them is divisible by 3. The former case is excluded by the assumption that $\gcd(x_n, y_n, z_n) = 1$ and the latter by $p|z_n$.

That's all. □

6. Let p be the line parallel to BC that passes through A and let $P = p \cap DG$. We denote the midpoint of BC by T.

We project the harmonic ratio $(BTC\infty) = -1$ from A onto the line DG and learn that $(EGFP) = -1$. Therefore by a well-known Apollonian property of harmonic ratios it suffices to prove $\angle PHA = 90°$.

Now let Q be the orthogonal projection of D onto AH. The homothety centered at G with factor $-\frac{1}{2}$ maps the line p onto the line BC and hence it maps P to D. Moreover, it leaves the line AH intact, so we just need to prove that it maps H to Q.

As H lies on the circumcircle of ABC, which is mapped to the nine-point circle ω of ABC by the considered homothety, we just need to verify that Q lies on ω and that Q is not the "wrong" intersection point of ω and AH. But this other point is the midpoint of BC, which does not coincide with Q when $AB \neq AC$.

So our current claim is just that Q lies on ω. To verify it, we denote the orthogonal projection of A to BC by R and the midpoint of AB by S. It is known that R, S, and T lie on ω. Further, the points Q and R lie on the circle with diameter AD. Hence

$$\begin{aligned}\angle RQT = \angle BDA = \beta - \angle BAD = \beta - \gamma = \alpha - (180^\circ - 2\beta) = \angle BST - \angle BSR \\ = \angle RST\end{aligned}$$

and we may conclude. $\square$