Czech-Polish-Slovak Mathematical Match before 58$^{\text{th}}$ IMO

IST Austria, Day 1, June 26$^{\text{th}}$, 2017

1. Find all positive real numbers c such that there are infinitely many pairs of positive integers (n, m) satisfying the following conditions: $n \geq m + c\sqrt{m-1} + 1$ and among numbers $n, n+1, \ldots, 2n-m$ there is no square of an integer.

2. Let ω be the circumcircle of an acute-angled triangle ABC. Point D lies on the arc BC of ω not containing point A. Point E lies in the interior of the triangle ABC, does not lie on the line AD, and satisfies $\angle DBE = \angle ACB$ and $\angle DCE = \angle ABC$. Let F be a point on the line AD such that lines EF and BC are parallel, and let G be a point on ω different from A such that $AF = FG$. Prove that points D, E, F, G lie on one circle.

3. Let k be a fixed positive integer. A finite sequence of integers $x_1, x_2, \ldots, x_n$ is written on a blackboard. Pepa and Geoff are playing a game that proceeds in rounds as follows.

- In each round, Pepa first partitions the sequence that is currently on the blackboard into two or more contiguous subsequences (that is, consisting of numbers appearing consecutively). However, if the number of these subsequences is larger than 2, then the sum of numbers in each of them has to be divisible by k.
- Then Geoff selects one of the subsequences that Pepa has formed and wipes all the other subsequences from the blackboard.

The game finishes once there is only one number left on the board. Prove that Pepa may choose his moves so that independently of the moves of Geoff, the game finishes after at most $3k$ rounds.

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4. Let ABC be a triangle. Line ℓ is parallel to BC and it respectively intersects side AB at point D, side AC at point E, and the circumcircle of the triangle ABC at points F and G, where points F, D, E, G lie in this order on ℓ. The circumcircles of triangles FEB and DGC intersect at points P and Q. Prove that points A, P, Q are collinear.

5. Each of the $4n^2$ unit squares of a $2n \times 2n$ board ($n \geq 1$) has been colored blue or red. A set of four different unit squares of the board is called *pretty* if these squares can be labeled A, B, C, D in such a way that A and B lie in the same row, C and D lie in the same row, A and C lie in the same column, B and D lie in the same column, A and D are blue, and B and C are red. Determine the largest possible number of different pretty sets on such a board.

6. Find all functions $f\colon (0, +\infty) \to \mathbb{R}$ satisfying

$$f(x) - f(x+y) = f\left(\frac{x}{y}\right) f(x+y) \qquad \text{for all } x, y > 0.$$