1. Find all positive real numbers c such that there are infinitely many pairs of positive integers (n, m) satisfying the following conditions: $n \geq m + c\sqrt{m-1} + 1$ and among numbers $n, n+1, \ldots, 2n-m$ there is no square of an integer.

(Slovakia)

Solution: We prove that c satisfies the condition in the statement if and only if $c \leq 2$.

Let us first consider any $c \leq 2$. For any positive integer k, define

$$n = k^2 + 1 \quad \text{and} \quad m = (k-1)^2 + 1$$

Observe that

$$m + c\sqrt{m-1} + 1 \leq k^2 - 2k + 2 + 2(k-1) + 1 = k^2 + 1 = n$$

and

$$(n, n+1, \ldots, 2n-m) = (k^2+1, k^2+2, \ldots, k^2+2k).$$

Therefore, every such pair (n, m) indeed satisfies the property from the problem statement, and there are infinitely many such pairs.

Now let us consider any $c > 2$, and let (n, m) be any pair of positive integers satisfying the property from the problem statement. Observe that for each positive integer n, the number $\lceil \sqrt{n} \rceil^2$ is always between numbers n and $(\sqrt{n}+1)^2$ (inclusive), hence there is always a square of an integer in the range

$$n, n+1, \ldots, n + \lfloor 2\sqrt{n} \rfloor + 1.$$

This implies that $2n - m < n + \lfloor 2\sqrt{n} \rfloor + 1$, so in particular

$$m \geq n - 2\sqrt{n}.$$

Combining this with the inequality from the problem statement yields

$$n \geq n - 2\sqrt{n} + c\sqrt{n - 2\sqrt{n} - 1} + 1. \tag{1}$$

Observe that since $c > 2$, we have $c\sqrt{n - 2\sqrt{n} - 1} > 2\sqrt{n}$ for large enough n. Indeed, equivalently we have $1 - \frac{2}{\sqrt{n}} - \frac{1}{n} > \frac{4}{c^2}$, and the left-hand side tends to 1 as n grows to infinity while the right hand side is strictly smaller than 1. This implies that (1) may be satisfied only for finitely many positive integers n. Since $m \leq n$ for all pairs (n, m) satisfying the conditions from the problem statement, this implies that there are only finitely many such pairs (n, m). $\square$

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2. Let ω be the circumcircle of an acute-angled triangle ABC. Point D lies on the arc BC of ω not containing point A. Point E lies in the interior of the triangle ABC, does not lie on the line AD, and satisfies $\angle DBE = \angle ACB$ and $\angle DCE = \angle ABC$. Let F be a point on the line AD such that lines EF and BC are parallel, and let G be a point on ω different from A such that $AF = FG$. Prove that points D, E, F, G lie on one circle.

(Slovakia)

Solution: Denote $\alpha = \angle CAB$, $\beta = \angle ABC$, and $\gamma = \angle BCA$. Let K and L be the second intersections of lines BE and CE with ω, respectively, different from B and C. Observe that

$$\angle BAK = \angle BAD + \angle DAK = \angle BAD + \angle DBE = \angle BAD + \gamma = \angle ACD$$

and symmetrically $\angle CAL = \angle ABD$. It follows that arcs AD, BK, and CL of ω have equal lengths, so chords AD, BK, and CL also have equal lengths. In particular, since $E = BK \cap CL$ does not lie on AD, these chords are not diameters. It follows that if O is the center of ω, then O does not lie on any of the chords AD, BK, CL, and in particular $O \neq E$. Moreover, O and E lie on the same side of line AD. Suppose without loss of generality that O and E lie in triangle ACD, for the second case is symmetric.

Observe that

$$\angle BEC = 360^\circ - \angle DBE - \angle DCE - \angle BDC = 360^\circ - \beta - \gamma - (180^\circ - \alpha) = 2\alpha = \angle BOC,$$

which implies that B, O, E, C are concyclic. Further, if we denote $P = AD \cap BC$, then we have

$$\begin{aligned} \angle DOE &= \angle BOE - \angle BOD = 180^\circ - \angle BCE - 2\angle BAD \\ &= 180^\circ - \angle DCE - \angle BAD = 180^\circ - \beta - \angle BAD = \angle APB = \angle EFD, \end{aligned}$$

which implies that D, F, O, E are also concyclic.

Consider triangles AFO and GFO. We have $AF = FG$ by assumption, also $OA = OG$ since G lies on ω, hence these two triangles are congruent. In particular $\angle FAO = \angle FGO$. Triangle AOD is isosceles, hence $\angle FAO = \angle FDO$. This implies that F, O, G, D are concyclic as well.

Since points F, O, D are pairwise distinct, this implies that all five points D, F, O, E, G lie on the circumcircle of triangle FOD, so in particular D, E, F, G are concyclic. □

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3. Let k be a fixed positive integer. A finite sequence of integers $x_1, x_2, \dots, x_n$ is written on a blackboard. Pepa and Geoff are playing a game that proceeds in rounds as follows.

- In each round, Pepa first partitions the sequence that is currently on the blackboard into two or more contiguous subsequences (that is, consisting of numbers appearing consecutively). However, if the number of these subsequences is larger than 2, then the sum of numbers in each of them has to be divisible by k.
- Then Geoff selects one of the subsequences that Pepa has formed and wipes all the other subsequences from the blackboard.

The game finishes once there is only one number left on the board. Prove that Pepa may choose his moves so that independently of the moves of Geoff, the game finishes after at most $3k$ rounds.

(Poland)

Solution: A finite sequence of integers is called a *word* and any its contiguous subsequence is called a *subword*. For a word u, by $\sum u$ we denote the sum of numbers in u. A *prefix* of a word is a subword starting at the beginnning of the word, and a prefix is *proper* if it is neither empty nor the whole word. Analogously we define suffixes.

For a word u, let $R(u) \subseteq \{0, 1 \dots, k-1\}$ be the set of remainders r modulo k for which there exists a proper prefix v of u with $\sum v \equiv r \mod k$. In other words, $R(u)$ comprises different remainders modulo k realized by sums of numbers in proper prefixes of u. Define the *rank* of u as $|R(u)|$.

We shall prove the following statement: given a word u on the board, Pepa can always play at most 3 rounds so that the rank of the remaining word is strictly smaller than the rank of u. Since the rank of the initial word is at most k and the rank of a word is 0 if and only if it consists of one number, in this way Pepa may force the end of the game within at most $3k$ rounds.

Assume then that the word u on the board has length larger than 1, and take any $r \in R(u)$. Suppose that the proper prefixes of u giving remainder r modulo k end at positions $1 \leq i_1 < i_2 < \dots < i_p < |u|$, where $p \geq 1$. Consider the following partition of u into subwords:

$$u = v_0 v_1 v_2 \dots v_{p-1} v_p,$$

where v_0 is the prefix up to position i_1, each v_j for $j = 1, 2, \dots, p-1$ is the subword between positions $i_j + 1$ and i_{j+1}, and v_p is the suffix from position $i_p + 1$ till the end of the word.

We observe that the rank of each of subword v_j is strictly smaller than the rank of u. For $j = 0$ this is trivial: since i_1 is the first position at which a prefix of u has sum congruent to r modulo k, we have that $R(v_0) \subseteq R(u) \setminus \{r\}$. For $j > 0$, take any proper prefix w of v_j, let $w' = v_0 v_1 \dots v_{j-1} w$, and let a be the remainder of $\sum v_0 v_1 \dots v_{j-1}$ modulo k. Observe that $\sum w' \equiv a + \sum w \mod k$. Therefore, the remainders realized by proper prefixes w of v_j are exactly the remainders realized by prefixes w' as above with a subtracted modulo k. Since between i_j and i_{j+1} there is no position at which a prefix of u has sum congruent to r modulo

k, we infer that none of prefixes w' as above has sum congruent to r modulo k. This implies that $R(v_j) \subseteq \{q - a\colon q \in R(u) \setminus \{r\}\}$, so $|R(v_j)| < |R(u)|$.

Note that $\sum v_j \equiv 0 \mod k$ for each $j = 1, 2, \dots, p-1$ by construction. All these observations lead to the following three-turn strategy for Pepa:

- Partition u into v_0 and $v_1 v_2 \dots v_p$. If Geoff chooses v_0, then the rank of the word has already decreased. Otherwise Geoff chooses $v_1 v_2 \dots v_p$.
- Partition $v_1 v_2 \dots v_p$ into $v_1 v_2 \dots v_{p-1}$ and v_p. If Geoff chooses v_p, then the rank of the word has already decreased. Otherwise Geoff chooses $v_1 v_2 \dots v_{p-1}$.
- Partition $v_1 v_2 \dots v_{p-1}$ into $v_1, v_2, \dots, v_{p-1}$, which are all words with sums of numbers divisible by k. Regardless of the move of Geoff, the rank of the word chosen by him is strictly smaller than the rank of u.

This concludes the proof. $\square$

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4. Let ABC be a triangle. Line ℓ is parallel to BC and it respectively intersects side AB at point D, side AC at point E, and the circumcircle of the triangle ABC at points F and G, where points F, D, E, G lie in this order on ℓ. The circumcircles of triangles FEB and DGC intersect at points P and Q. Prove that points A, P, Q are collinear.

(Slovakia)

Solution: Let ω_B, ω_C be the circumcircles of triangles FEB and DGC respectively. Since PQ is the radical axis of ω_B and ω_C, it is sufficient to prove that the powers of A with respect to ω_B and ω_C are equal. If we denote by X the second intersection of ω_B and AC, and by Y the second intersection of ω_C and AB, then this is equivalent to

$$AX \cdot AE = AY \cdot AD.$$

Lines DE and BC are parallel, yielding $\frac{AB}{AD} = \frac{AC}{AE}$, so the above is equivalent to

$$AX \cdot AC = AY \cdot AB.$$

This, in turn, is equivalent to B, Y, X, C being concyclic. From angles in ω_B and ω_C we have

$$\angle BYC = \angle DYC = \angle DGC \quad \text{and} \quad \angle BXC = \angle BXE = \angle BFE.$$

On the other hand, trapezoid $BFGC$ is inscribed in the circumcircle of ABC, so it is isosceles, hence $\angle BFE = \angle DGC$. We conclude that $\angle BYC = \angle BXC$, so B, Y, X, C are indeed concyclic and we are done. $\square$

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5. Each of the $4n^2$ unit squares of a $2n \times 2n$ board ($n \geq 1$) has been colored blue or red. A set of four different unit squares of the board is called *pretty* if these squares can be labeled A, B, C, D in such a way that A and B lie in the same row, C and D lie in the same row, A and C lie in the same column, B and D lie in the same column, A and D are blue, and B and C are red. Determine the largest possible number of different pretty sets on such a board.

(Poland)

Solution: Let us index the unit squares of the board by pairs of integers (a, b) with $1 \leq a, b \leq 2n$. We prove that the largest possible number of pretty sets is n^4. For the upper bound, consider coloring all the unit squares (a, b) with $a, b \leq n$ or $a, b \geq n+1$ blue, and all the other unit squares red. It is straightforward to verify that this coloring yields n^4 pretty sets. Thus we are left with proving that no coloring yields more pretty sets.

Call an unordered pair of distinct unit squares $\{A, B\}$ *mixed* if A and B are in the same row and A and B have different colors. Clearly, if a row contains a blue squares and b red squares ($a + b = 2n$), then it contains $ab \leq n^2$ mixed pairs. Therefore, there are at most $2n^3$ mixed pairs in total.

Let every mixed pair $\{A, B\}$ *charge* the unordered pair $\{i, j\}$ of distinct columns such that A is in column i and B is in column j. Denote by $\mathsf{charge}(i, j)$ the number of times the pair of columns $\{i, j\}$ is charged.

Obviously, every pair of columns is charged at most $2n$ times, i.e., $\mathsf{charge}(i,j) \leq 2n$. Moreover, since there are at most $2n^3$ mixed pairs in total, we have $\sum_{\{i,j\}} \mathsf{charge}(i,j) \leq 2n^3$ where the summation is over pairs of distinct columns $\{i,j\}$.

Observe that if a pair of distinct columns $\{i,j\}$ is charged $k = \mathsf{charge}(i,j)$ times, then there are at most $\frac{k^2}{4}$ pretty sets with squares contained in these columns. This is because for some a, b with $a+b=k$ there are a red-blue and b blue-red mixed pairs within these columns, yielding $ab \leq \frac{k^2}{4}$ pretty sets. Therefore, the total number of pretty sets is at most

$$\frac{1}{4}\sum_{\{i,j\}} \mathsf{charge}(i,j)^2 \leq \frac{n}{2} \cdot \sum_{\{i,j\}} \mathsf{charge}(i,j) \leq n^4,$$

where again the summation is over unordered pairs of distinct columns. This concludes the proof. $\square$

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6. Find all functions $f\colon (0,+\infty) \to \mathbb{R}$ satisfying

$$f(x) - f(x+y) = f\left(\frac{x}{y}\right) f(x+y) \qquad \text{for all } x, y > 0.$$

(Austria)

Solution: Suppose $f(t) = 0$ for some $t > 0$. For $0 < x < t$ we choose $y = t - x > 0$ and find $f(x) = 0$. From setting $x = y = 1$ we conclude that $f(1) \neq -1$. Hence by setting $x = y$ we get $f(x) - f(2x) = f(1)f(2x)$ for $x > 0$. Inductively we find

$$f(2^n x) = f(x)(1 + f(1))^{-n}. \tag{2}$$

Hence for an arbitrary $x > 0$ we choose $n \in \mathbb{N}$ such that $\frac{x}{2^n} \leq t$ and we conclude from (2) that $f(x) = 0$.

Now suppose $f(t) \neq 0$ for all $t > 0$. We define $g(x) := f(x)^{-1}$, $x > 0$, and rewrite the given equation as

$$g(x+y) - g(x) = \frac{g(x)}{g\left(\frac{x}{y}\right)} \qquad \text{for all } x,\, y > 0. \tag{3}$$

By setting $y = 1$ we obtain $g(x+1) = g(x) + 1$ for all $x > 0$. It follows that $g(n) = n - 1 + g(1)$ holds for all $n \in \mathbb{N}$. Setting $x = y = 2$ in (3) we now find $g(1) = 1$. Setting $x = 1$ we obtain $g(y)g\left(\frac{1}{y}\right) = 1$ for all $y > 0$. We can now rewrite (3) as

$$g(x+y) = g(x) + g(x)g\left(\tfrac{y}{x}\right) = g(x)\left(1 + g\left(\tfrac{y}{x}\right)\right) = g(x)g\left(\tfrac{x+y}{x}\right). \tag{4}$$

From (4) we directly see that $g(a)g(b) = g(ab)$ for all $a > 0$ and $b > 1$. Using $g(y)g\left(\frac{1}{y}\right) = 1$, we can rewrite (4) as

$$g\left(\tfrac{x}{x+y}\right)g(x+y) = g(x),$$

which together with the previous line shows that $g(a)g(b) = g(ab)$ holds indeed for all $a,\, b > 0$. In particular we see that $g(a) = g(\sqrt{a})^2 > 0$, so g is a positive function. Also, from the first equation in (4) we now infer the functional equation

$$g(x+y) = g(x) + g(y) \quad \text{for all } x, y \in (0, \infty).$$

It is well known that this implies $g(x) = xg(1) = x$ for $x \in \mathbb{Q}_{>0}$. Since g is positive, by (3) we deduce that $g(x+y) > g(x)$, so g is strictly increasing. Hence $g(x) = x$ for all $x > 0$.

Since $f(x) = 0$ and $f(x) = \frac{1}{x}$ obviously satisfy the given equation, we have found all solutions. $\square$