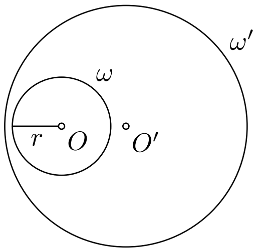
4. *Let ABC be an acute triangle with the perimeter of $2s$. We are given three pairwise disjoint circles with pairwise disjoint interiors with the centres A, B and C, respectively. Prove that there exists a circle with the radius of s which contains all the three circles.* (*Josef Tkadlec, Czechia*)

Solution. To simplify the formulations, we say that a point lies inside of the circle if it lies on that circle or in its interior. Assume we are given a circle ω with the radius of r and the centre O. A circle ω' with the centre O' contains the circle ω if and only if its radius is at least $O'O + r$.

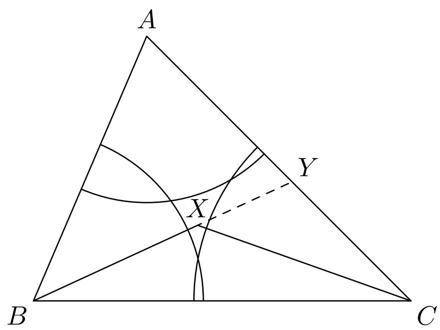


Denote by r_a, r_b, r_c the radii of our circles with the centres at A, B and C, respectively. Using our observation three times indicates that the centre X of the circle we are seeking has to meet $s \geq AX + r_a$, or equivalently $AX \leq s - r_a$, and analogously $BX \leq s - r_b$ and $CX \leq s - r_c$.

Notice that the numbers $s - r_a$, $s - r_b$ and $s - r_c$ are positive. We will show this for $s - r_a$. Since our circles are disjoint with disjoint interiors, we know that $r_a < b$ and $r_a < c$. This gives us $r_a < (b + c)/2 < (a + b + c)/2 = s$, which indeed means that $s - r_a$ is a positive number.

Now we may consider three circles with the centres A, B and C and radii $s - r_a$, $s - r_b$ and $s - r_c$, respectively. If we prove that there is a point X lying inside each of them, we will be done.

Each two of these three circles intersect at two points, because for example $(s - r_a) + (s - r_b) > 2s - c = a + b > c$ (and also $c > |(s - r_a) - (s - r_b)|$). For the sake of contradiction assume there is no point lying inside all of them. Then the situation looks like on the picture, that is, there exists a point X inside of the triangle which lies outside of the three circles (see the remark at the end):

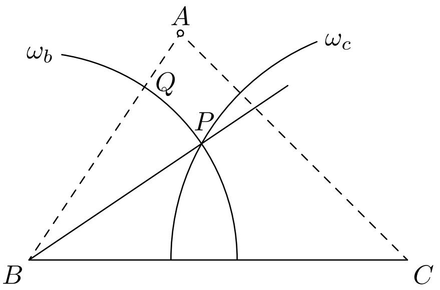


For such X we have $AX + BX + CX > s - r_a + s - r_b + s - r_c > 2s$. This is not possible, however. Let Y be the intersection of BX and AC. Then usign the triangle inequalities for the triangles CXY, ABY we get

$$BX + CX < BX + XY + CY = BY + CY < AB + AY + CY = AB + AC.$$

Similarly $AX + BX < AC + BC$ and $CX + AX < BC + AB$. Summing these three inequalities we obtain $AX + BX + CY < AB + BC + AC = 2s$, which is a contradiction.

Remark. If three circles ω_a, ω_b and ω_c with the centres A, B and C, respectively, satisfy the conditions that each two of them intersect and there is no point lying inside all of the three circles, then there exists a point in the interior of the triangle ABC which lies outside of each of the three circles.



To prove this, consider the intersection point P of ω_b and ω_c which lies in the halfplane determined by the line BC and the point A. The intersection Q of the ray BA with ω_b lies inside of ω_a, since it is the closest point of ω_b to A (this is true even if A is inside of ω_b, since $\omega_a \cap \omega_b \neq \emptyset$). Therefore A cannot lie in the angle CBP (otherwise Q would lie inside all of the three circles). But that means P lies in the interior of the angle CBA. Similarly P lies in the interior of the angle BCA. So we have that P lies in the interior of the triangle ABC. Since P does not lie inside of ω_a, there is a point in the neigbourhood of P lying outside of all the three circles.

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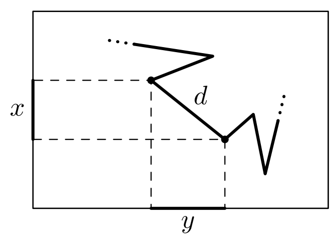
5. *In a rectangle with dimensions* 2×3 *there is a polyline of length 36, which can have self-intersections. Show that there exists a line parallel to two sides of the rectangle, which intersects the other two sides in their interior points and intersects the polyline in fewer than 10 points.*

(*Josef Tkadlec, Czechia, Vojtech Bálint, Slovakia*)



Solution. Consider an arbitrary line segment of the polyline and denote by d its length and by x and y the lengths of its perpendicular projections on the sides of lengths 2 and 3, respectively. Cauchy-Swartz inequality gives us

$$(2x + 3y)^2 \leq (2^2 + 3^2)(x^2 + y^2) = 13d^2,$$

which means $2x + 3y \leq d \cdot \sqrt{13}$. Denote by X and Y the total length of all the perpendicular projections of all the line segments on the sides of lengths 2 and 3, respectively. Summing up our estimations for each line segment gives us $2X+3Y \leq 36 \cdot \sqrt{13} < 130$. But then either $2X < 40$, or $3Y < 90$. In the first case we would have $X < 20$, so on the side of length 2 there is a point that is contained in fewer than 10 projections. A line perpendicular to this side at this point intersects the polyline at most 9 times. The other case is analogous.

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6. *We say that a positive integer n is fantastic, if there exist positive rational numbers a and b such that*

$$n = a + \frac{1}{a} + b + \frac{1}{b}.$$

(a) Prove that there exist infinitely many prime numbers p such that no multiple of p is fantastic.

(b) Prove that there exist infinitely many prime numbers p such that some multiple of p is fantastic.

(*Walther Janous, Austria*)

Solution. Note that

$$r(a,b) := a + \frac{1}{a} + b + \frac{1}{b} = \frac{(a+b)(ab+1)}{ab}.$$

We put $a = \frac{t}{u}$ and $b = \frac{v}{w}$, where t, u, v and w are positive integers such that both t and u and also v and w are coprime. Then we get $r(a,b) = \frac{(tv+uw)(tw+uv)}{tuvw}$, whence the Diophantine equation

$$tu(v^2+w^2) + vw(t^2+u^2) = kptuvw \tag{6}$$

has to be investigated. Now $\gcd(tu, t^2+u^2) = 1$. Therefore, (6) implies $tu \mid vw$. As we get similarly $vw \mid tu$, too, we infer

$$tu = vw \tag{7}$$

and (6) becomes

$$\frac{(v^2+t^2)(v^2+u^2)}{v^2} = t^2+u^2+v^2+w^2 = kptu.$$

Therefore, p has to divide either v^2+t^2 or v^2+u^2. In the case $p \equiv -1 \pmod 4$, i. e. when -1 is a quadratic non-residue mod p, this means that p divides v (and t or u). But since the same argument is valid for w instead of v, we have $p \mid v, w$ contradicting the coprimality of v and w. Thus the infinitely many primes with $p \equiv -1 \pmod 4$ have no fantastic multiple and part (a) is solved.

For part (b) we choose $v = 1$ and substitute $w = tu$. Thus we are looking for integers t and u such that

$$1 + t^2 + u^2 + t^2u^2 = kptu.$$

Here we choose[1] $t = F_{2l+1}$, $u = F_{2l-1}$ and use the identity[2] $1 + F_{2l+1}^2 = F_{2l+3}F_{2l-1}$ to obtain

$$(1+t^2)(1+u^2) = (1+F_{2l+1}^2)(1+F_{2l-1}^2) = F_{2l+3}F_{2l-1}F_{2l+1}F_{2l-3} = kptu = kpF_{2l+1}F_{2l-1},$$

i. e. $F_{2l+3}F_{2l-3} = kp$. Therefore every prime factor of the Fibonacci number F_{2l+3} has a fantastic multiple.

In view of the well-known formula $\gcd(F_a, F_b) = F_{\gcd(a,b)}$ it is clear that F_a and F_b are relatively prime, if a and b are different prime numbers. Hence we know that infinitely many prime numbers have a fantastic multiple, which solves part (b).

[1] It is a well-known problem that $tu \mid t^2 + u^2 + 1$ with $t > u$ is only possible if t and u are Fibonacci numbers of the form $t = F_{2l+1}$, $u = F_{2l-1}$ in which case $t^2 + u^2 + 1 = 3tu$.

[2] This is a special case of Vajda's identity $F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_iF_j$