

Czech-Polish-Slovak Match

IST Austria, 24–27 June 2018

1. Determine all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that, for all real numbers x and y ,

$$f(x^2 + xy) = f(x)f(y) + yf(x) + xf(x + y).$$

(Walther Janous, Austria)

Solution. Setting $x = 0$ yields $f(0) = f(0)f(y) + yf(0)$. If $f(0) \neq 0$, we obtain $1 = f(y) + y$ or equivalently $f(y) = 1 - y$ for all $y \in \mathbb{R}$. Inserting this in the original equation yields

$$1 - x^2 - xy = (1 - x)(1 - y) + y(1 - x) + x(1 - x - y)$$

for $x, y \in \mathbb{R}$, which is true.

Therefore, we are left with $f(0) = 0$.

With the substitution $x + y = z$, the functional equation is equivalent to

$$f(xz) = f(x)f(z - x) + (z - x)f(x) + xf(z). \quad (1)$$

Exchanging x and z yields

$$f(zx) = f(z)f(x - z) + (x - z)f(z) + zf(x). \quad (2)$$

Combining (1) and (2) yields

$$f(x)f(z - x) + (z - x)f(x) + xf(z) = f(z)f(x - z) + (x - z)f(z) + zf(x)$$

or equivalently

$$f(x)(f(z - x) - x) = f(z)(f(x - z) - z). \quad (3)$$

Setting $z = 0$ yields

$$f(x)(f(-x) - x) = f(0)f(x) = 0,$$

so for each $x \in \mathbb{R}$ we either have $f(x) = 0$ or $f(-x) = x$.

Assume that there is an $x \neq 0$ with $f(x) = 0$. For $z \neq 0$, we have $f(x - z) \in \{0, z - x\}$, so $f(x - z) \neq z$. Thus (3) implies $f(z) = 0$ for all $z \neq 0$. It is clear that the constant function $f = 0$ is a solution.

Otherwise, we have $f(x) \neq 0$ for all $x \neq 0$ and therefore $f(x) = -x$ for all x . This is also a solution.

We conclude that there are three solutions f , namely $f = 0$, $f(x) = -x$ and $f(x) = 1 - x$.

Alternative Solution. Setting $x = 0$ gives $f(0) = f(0)f(y) + yf(0)$. Since $f(y) = 1 - y$ gives a solution, we remain with the case $f(0) = 0$. Setting $x = 1$ gives

$$f(1 + y) = f(1)f(y) + yf(1) + f(1 + y), \quad \text{i. e.} \quad 0 = f(1)(f(y) + y).$$

Since $f(y) = -y$ is a solution, we remain with the case $f(1) = 0$.

Now suppose $f(0) = f(1) = 0$. Setting $y = -x$ gives $0 = f(x)f(-x) - xf(x)$, that is

$$f(x) = 0 \quad \vee \quad f(-x) = x. \quad (4)$$

Setting $y = 1 - x$ gives $f(x) = f(x)f(1 - x) + (1 - x)f(x)$, that is

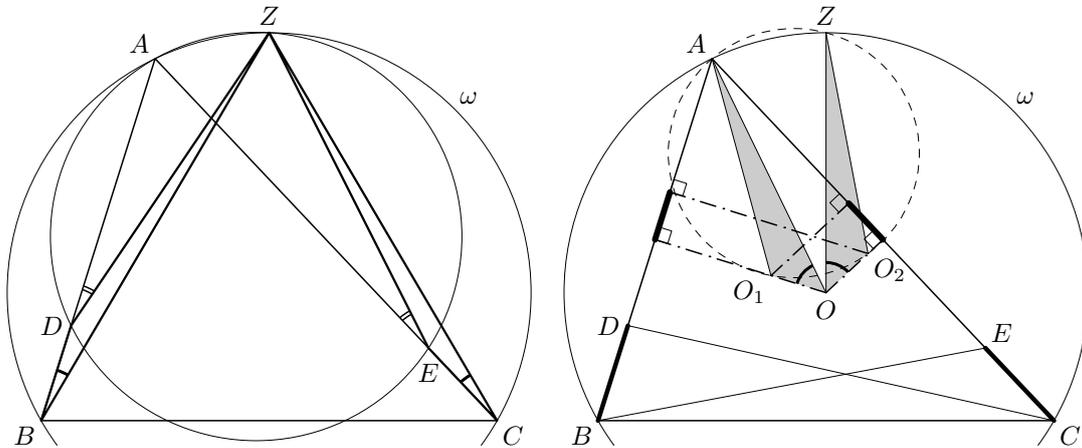
$$f(x) = 0 \quad \vee \quad f(1 - x) = x. \quad (5)$$

If $f(x) \neq 0$ for some $x \neq 0$, we have $f(-x) = x \neq 0$ and $f(1 - x) = x \neq 0$ and applying (4) and (5) to $-x$ and $1 - x$ instead of x we get $f(x) = -x = 1 - x$, a contradiction. Hence $f(x) = 0$ for all $x \in \mathbb{R}$.

2. Let ABC be an acute scalene triangle. Let D and E be points on the sides AB and AC , respectively, such that $BD = CE$. Denote by O_1 and O_2 the circumcentres of the triangles ABE and ACD , respectively. Prove that the circumcircles of the triangles ABC , ADE and AO_1O_2 have a common point different from A .

(Patrik Bak, Slovakia)

Solution 1. Let Z be the midpoint of the longer arc BC of the circumcircle ω of the triangle ABC . The triangles ZDB and ZEC are congruent, because they agree in the sides $BD = CE$ and $ZB = ZC$, as well as in the corresponding angles between them, for both lie over the chord AZ of ω . It follows that $\angle ZDA = \angle ZEA$, which in turn discloses that the quadrilateral $ADEZ$ is cyclic. So it remains to be shown that the quadrilateral AO_1O_2Z is cyclic.

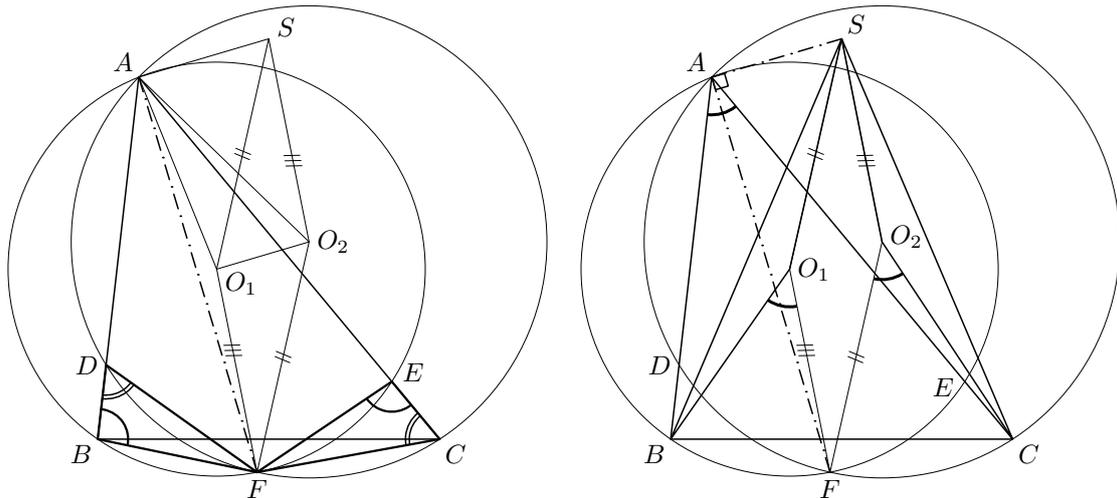


The center O of ω satisfies $OO_1 \perp AB$ and $OO_2 \perp AC$. The projections of O and O_1 onto AC are the midpoints of AC and AE respectively. Thus the projection of the segment OO_1 onto AC has length $\frac{1}{2}CE$. For the same reason, the projection of OO_2 on AB has length $\frac{1}{2}BD$, and by hypothesis these two length agree. Moreover, the angle between OO_1 and AC is the same as the angle between OO_2 and AB . It follows that $OO_1 = OO_2$.

Further, we have $\angle AOO_1 = \angle ACB = \angle O_2OZ$, the latter being a consequence of $ZO \perp BC$ and $OO_2 \perp AC$. So the rays OA and OZ are isogonal in the angle O_2OO_1 . In the combination with $AO = ZO$ and $OO_1 = OO_2$ this proves that

the quadrilateral AO_1O_2Z is an isocleses trapezium and thus in particular cyclic. Thereby the problem is solved.

Solution 2. Let the circumcircles of triangles ABE and ADC intersect each other again at $F \neq A$. Then the triangles BFD and EFC are congruent, for they agree in their sides $BD = CE$ as well as in their corresponding adjacent angles, i. e., $\angle FBD = \angle FEC$ and $\angle BDF = \angle ECF$. It follows that the altitudes of these triangles passing through F have the same lengths, wherefore AF is the bisector of the angle BAC .



Now construct the point S such that SO_1FO_2 is a parallelogram. We will show that S is the desired point.

To prove that S lies on the circumcircle of triangle AO_1O_2 , we note that the triangles AO_1O_2 and FO_1O_2 are congruent due to $AO_1 = FO_1$ and $AO_2 = FO_2$. It follows that AO_1O_2S is an isosceles trapezium and hence in particular a cyclic quadrilateral, as claimed. Later, it will help us to have observed that the facts used in this paragraph imply $AS \parallel O_1O_2 \perp AF$.

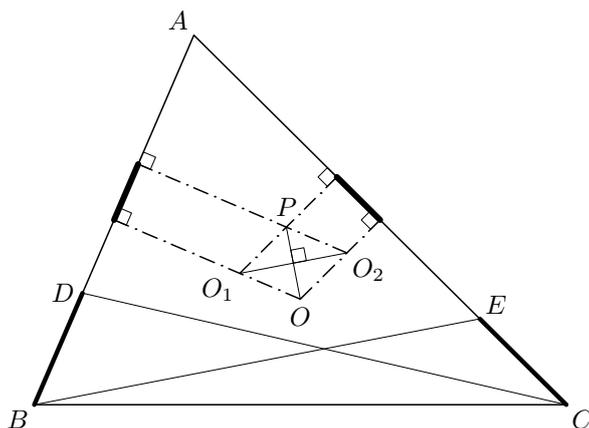
Next, we prove that S lies on the circumcircle of the triangle ABC and that it is actually the midpoint of its longer arc BC ; this will also show $S \neq A$, as needed. Our first intermediate step is to observe that the triangles O_1SB and O_2CS are congruent. Indeed they agree in a pair of sides, $BO_1 = FO_1 = SO_2$ and $SO_1 = FO_2 = CO_2$. Moreover the corresponding angles between these sides are equal, because their complements to 360° are equal as a consequence of

$$\angle BO_1F = 2\angle BAF = 2\angle FAC = \angle FO_2C$$

and $\angle FO_1S = \angle SO_2F$. This concludes the verification of $\triangle O_1SB \cong \triangle O_2CS$, and it follows that $BS = CS$. Further, since AF is the bisector of $\angle BAC$ and $AF \perp AS$, the line AS is the exterior bisector of $\angle BAC$. Altogether we obtain that S is the point described above. The fact that S lies on the circumcircle of the triangle ADE can be shown as in the first solution.

Solution 3. Denote by O and P the circumcentres of triangles ABC and ADE , respectively. The lines OO_1 and O_2P (being the perpendicular bisectors of AB

and AD , respectively) are both perpendicular to AB and their distance is $\frac{1}{2}BD$. Similarly, the lines OO_2 and O_1P are both perpendicular to AC and their distance is $\frac{1}{2}CE$. Since $\frac{1}{2}BD = \frac{1}{2}CE$, the quadrilateral O_1OO_2P is a parallelogram with equal altitudes, hence a rhombus. It follows that OP is the perpendicular bisector of O_1O_2 , so all the three circumcentres of the triangles ABC , ADE and AO_1O_2 lie on the same line, which concludes the claim (since A does not lie on this line because of $AB \neq AC$).



3. *There are 2018 players sitting around a round table. At the beginning of the game we arbitrarily deal all the cards from a deck of K cards to the players (some players may receive no cards). In each turn we choose a player who draws one card from each of the two neighbours. It is only allowed to choose a player whose each neighbour holds a nonzero number of cards. The game terminates when there is no such player. Determine the largest possible value of K such that, no matter how we deal the cards and how we choose the players, the game always terminates after a finite number of turns.* (Peter Novotný, Slovakia)

Solution. The answer is $K = 2017$.

For $K = 2018$, we deal 2 cards to one player, 0 cards to one of his neighbours and 1 card to everyone else. Then in each turn we choose the player with 0 cards:

$$\dots 11 \widehat{201} 11 \dots \quad \rightarrow \quad \dots 11 \underline{120} 11 \dots$$

After each turn, the configuration stays the same – there is one player with 2 cards, one of his neighbours with 0 cards and all the others with 1 card (the only change is that the positions of the players with 2 and 0 cards is shifted). Therefore we can make moves forever and the game never terminates.

Whenever $K > 2018$, we can play forever using the same strategy as for $K = 2018$. We simply deal the extra cards arbitrarily and ignore them during the game.

Now we will prove that for $K = 2017$ the game terminates no matter how we play. Let us call *zeros* the players with no cards and *ones* the players with exactly one card. The zeros split the other players into *segments* of various lengths. When two zeros sit next to each other, they form a segment with a length of 0. Also note that there is obviously at least one zero when $K = 2017$.

Lemma. There exists a segment containing no other players than ones (possibly with a length of 0).

Proof. If we add to each segment the zero which bounds it in the clock-wise direction, then the sum of the lengths of all the segments will be 2018. There are only 2017 cards, therefore at least one segment contains fewer cards than players, which is possible only when all the players of this segment, except for the bounding zero, are ones. \square

Let us consider the shortest segment among the ones containing no other players than ones; the lemma assures the existence of such a segment. If we choose a zero adjacent to this segment, we shorten it by 1 (or by 2 — in the special case when there is exactly one zero in the game):

$$\dots * \widehat{0} \widehat{1} 1 1 \dots 1 0 \dots \quad \rightarrow \quad \dots * \underline{2} 0 1 1 \dots 1 0 \dots$$

If we choose one of the ones inside of the shortest segment, we create two even shorter segments:

$$\dots 0 1 \dots 1 \widehat{1} \widehat{1} 1 \dots 1 0 \dots \quad \rightarrow \quad \dots 0 1 \dots 1 \underline{0} \underline{3} 0 1 \dots 1 0 \dots \quad \rightarrow$$

The length of the shortest segment could decrease only finitely many times. From the moment when it stops decreasing we won't be able to choose any of the zeros bounding the shortest segment, nor any of the ones inside of it. This means that the game will continue on the other side of the table between the bounding zeros of the shortest segment. The neighbours of these two zeros won't be able to get any more cards, so we cannot choose them anymore. The neighbours of these neighbours will thereby be chosen at most finitely many times (at most the number of times equal to the number of cards of these neighbours), so after some time we won't be able to choose them. We can use this reasoning repeatedly. The part of the table where we still can choose players eventually decreases, which means that the game cannot last infinitely long.

Remark. If $K = 2018$ and we give one card to every player, then after one move we would get a segment of ones bounded by two zeros. In that case the game necessarily ends after finitely many moves (to see it we just need to use the reasoning from the solution).

Remark. As soon as we show that the game will be played only in one part of the table bounded by two players (so no cards will ever pass some line of the table and therefore it could be think of as a line segment), we might just use a right mono-variant to prove that the game is finite. For example, to each card we might assign its distance to one of the bounds and keep track of the sum of squares of these distances. In each move this number is decreased by

$$(a - 1)^2 + (a + 1)^2 - 2a^2 = 2,$$

and since it cannot be negative, the game will have to eventually end.