2007

56th Czech and Slovak Mathematical Olympiad

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First Round of the 56th Czech and Slovak Mathematical Olympiad
Problems for the take-home part
(October 2006)

1. *Find all real roots of the equation*

$$4x^4 - 12x^3 - 7x^2 + 22x + 14 = 0,$$

if it is known that it has four distinct real roots, two of which sum up to 1.

Solution. Denote the roots by x_1, x_2, x_3, x_4 in such a way that $x_1 + x_2 = 1$. Then

$$4x^4 - 12x^3 - 7x^2 + 22x + 14 = 4(x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

Comparing the coefficients at the corresponding powers of x, we obtain the familiar Vièta's relations

$$x_1 + x_2 + x_3 + x_4 = 3, \tag{1}$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = -\frac{7}{4}, \tag{2}$$

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -\frac{11}{2}, \tag{3}$$

$$x_1x_2x_3x_4 = \frac{7}{2}. \tag{4}$$

Since $x_1 + x_2 = 1$, it follows from (1) that $x_3 + x_4 = 2$. We rewrite the equations (2) and (3) in the form

$$\begin{aligned} (x_1 + x_2)(x_3 + x_4) + x_1x_2 + x_3x_4 &= -\frac{7}{4}, \\ (x_1 + x_2)x_3x_4 + (x_3 + x_4)x_1x_2 &= -\frac{11}{2}, \end{aligned}.$$

Upon substituting $x_1 + x_2 = 1$ and $x_3 + x_4 = 2$, this yields

$$\begin{aligned} x_1x_2 + x_3x_4 &= -\frac{15}{4}, \\ 2x_1x_2 + x_3x_4 &= -\frac{11}{2}. \end{aligned}$$

From this system of linear equations it is already easy to obtain

$$x_1 x_2 = -\frac{7}{4}, \qquad x_3 x_4 = -2.$$

Observe that for these values of the products x_1x_2 and x_3x_4, the equation (4) — which we have not used so far — is also satisfied. From the conditions $x_1 + x_2 = 1$, $x_1x_2 = -\frac{7}{4}$ it follows that x_1 and x_2 are the roots of the quadratic equation

$$x^2 - x - \frac{7}{4} = 0, \qquad \text{i.e. } x_{1,2} = \frac{1}{2} \pm \sqrt{2}.$$

Similarly, from the conditions $x_3 + x_4 = 2$ and $x_3x_4 = -2$ we obtain

$$x_{3,4} = 1 \pm \sqrt{3}.$$

Since, as we have already remarked, these roots satisfy all equations (1) to (4), they are also a solution to the original problem.

Conclusion. The roots of the equation are $\frac{1}{2} + \sqrt{2}$, $\frac{1}{2} - \sqrt{2}$, $1 + \sqrt{3}$, and $1 - \sqrt{3}$.

Other solution. From the hypothesis it follows that the left-hand side of the equation is the product of polynomials

$$x^2 - x + p \qquad \text{and} \qquad 4x^2 + qx + r,$$

where p, q and r are real numbers. Upon multiplying out and comparing coefficients at the corresponding powers of x, we obtain a system of four equations with three unknowns

$$\begin{aligned} q - 4 &= -12, \\ 4p - q + r &= -7, \\ pq - r &= 22, \\ pr &= 14. \end{aligned}$$

The first three equations have a unique solution $r = -8$, $p = -\frac{7}{4}$ and $q = -8$, which also fulfills the fourth equation. Thus we arrive at the decomposition

$$4x^4 - 12x^3 - 7x^2 + 22x + 14 = \left(x^2 - x - \frac{7}{4}\right)(4x^2 - 8x - 8).$$

The equation $x^2 - x - \frac{7}{4} = 0$ has roots $\frac{1}{2} \pm \sqrt{2}$, and the equation $4x^2 - 8x - 8 = 0$ has roots $1 \pm \sqrt{3}$.

2. *The incircle of a given triangle ABC touches its sides BC, CA, and AB at points K, L and M, respectively. Denote by P the intersection of the bisector of the interior angle at the vertex C with the line MK. Show that the lines AP and LK are parallel.*

Solution. Denote by k the incircle of the triangle ABC and by S its center. Let further α, β and γ denote the magnitudes of the interior angles in the triangle ABC in the usual way. Since the points K and L are axially symmetric with respect to the bisector of the interior angle at the vertex C, the lines KL and CP are perpendicular and $|\angle LPC| = |\angle KPC|$ (Fig. 1).

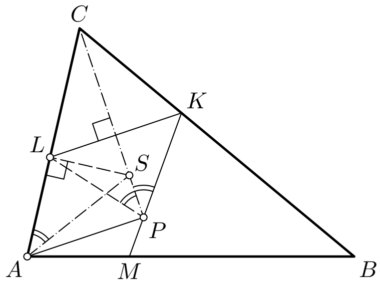

Fig. 1

Expressing the magnitudes of the interior angles at the bases KM and LK in the isosceles triangles KMB and LKC, respectively, we get $|\angle MKB| = 90° - \frac{1}{2}\beta$ and $|\angle LKC| = 90° - \frac{1}{2}\gamma$. Thus $|\angle MKL| = 90° - \frac{1}{2}\alpha$. Similarly it follows that $|\angle KLM| = 90° - \frac{1}{2}\beta$ and $|\angle LMK| = 90° - \frac{1}{2}\gamma$.

Since $|\angle KPC| + \frac{1}{2}\gamma = |\angle BKP| = 90° - \frac{1}{2}\beta$, we obtain the equality for the magnitude of the axially symmetric angles LPC and KPC

$$|\angle LPC| = |\angle KPC| = 90° - \frac{\beta + \gamma}{2} = \frac{\alpha}{2}.$$

The incircle k of the triangle ABC is at the same time the circumcircle of the triangle KLM, which is, in view of the magnitudes of its angles that we have computed, acute. The center S of this circle is therefore an interior point of the latter triangle, hence, an interior point of the segment CP. Since

$$|\angle LPC| = |\angle LPS| = |\angle LAS| = \frac{\alpha}{2},$$

the quadrangle $APSL$ is chordal. Since the angle ALS is right, the angle APS is also right (the lines AP and CP are perpendicular), thus the lines KL and AP are parallel. This completes the proof.

Remark. Since k is the circumcircle of the triangle KLM, it is easy to express its interior angles from the corresponding central angles: $|\angle KSL| = 180° - \gamma$, whence $|\angle KML| = 90° - \frac{1}{2}\gamma$, etc.

3. *If x, y, z are real numbers from the interval $\langle -1, 1\rangle$ such that $xy + yz + zx = 1$, then*

$$6\sqrt[3]{(1-x^2)(1-y^2)(1-z^2)} \leqslant 1 + (x+y+z)^2.$$

Give a proof, and find when the equality holds.

Solution. For any real numbers $x, y, z \in \langle -1, 1\rangle$, we have $1 - x^2 \geqslant 0$, $1 - y^2 \geqslant 0$, $1 - z^2 \geqslant 0$. Applying the inequality between the arithmetic and the geometric mean to the triple of nonnegative real numbers $1 - x^2$, $1 - y^2$, $1 - z^2$, we thus get

$$\sqrt[3]{(1-x^2)(1-y^2)(1-z^2)} \leqslant \frac{(1-x^2)+(1-y^2)+(1-z^2)}{3} = \frac{3-(x^2+y^2+z^2)}{3},$$

whence

$$6\sqrt[3]{(1-x^2)(1-y^2)(1-z^2)} \leqslant 6 - 2(x^2+y^2+z^2). \tag{1}$$

We show that if the real numbers $x, y, z \in \langle -1, 1\rangle$ satisfy $xy + yz + zx = 1$, then they also satisfy the inequality

$$6 - 2(x^2+y^2+z^2) \leqslant 1 + (x+y+z)^2. \tag{2}$$

Indeed, the right-hand side of this inequality has the form

$$1 + x^2 + y^2 + z^2 + 2(xy + yz + zx) = 3 + (x^2+y^2+z^2),$$

which upon substituting (2) leads to the equivalent inequality

$$x^2 + y^2 + z^2 \geqslant 1.$$

However, this is easily verified to be true: indeed, it suffices to show that for any real numbers x, y, z satisfying the hypothesis of our problem, we have the inequality

$$x^2 + y^2 + z^2 \geqslant xy + yz + zx,$$

which, however, is equivalent to the inequality

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \geqslant 0,$$

which holds for arbitrary real numbers x, y, z.

Conclusion. The inequality we were to prove follows from the inequalities (1) and (2). Equality takes place if and only if it takes places simultaneously in both (1) and (2); this happens if and only if $x = y = z$, which in view of the condition $xy + yz + zx = 1$ gives the only two solutions $x = y = z = \pm\frac{1}{3}\sqrt{3}$.

4. *Find for which natural numbers n it is possible to decompose the set $M = \{1, 2, \dots, n\}$ into a) two b) three mutually disjoint subsets having the same number of elements and such that each of them also contains the arithmetic mean of all its elements.*

Solution. a) Denote the desired subsets by A and B. Since they both have the same number of elements, the number of elements of M must be even. Thus $n = 2k$, where k is a natural number.

For $n = 4$ no such decomposition of $M = \{1, 2, 3, 4\}$ into two subsets can exist, since the arithmetic mean of two distinct numbers cannot be equal to either of these numbers. Let us construct a desired decomposition of the set M for the first few even values of the number n (the arithmetic mean of the elements in the subsets is set in boldface).

$n = 2$:	$A = \{\mathbf{1}\}$	$B = \{\mathbf{2}\}$
$n = 4$:	decomposition does not exist	
$n = 6$:	$A = \{1, \mathbf{2}, 3\}$	$B = \{4, \mathbf{5}, 6\}$
$n = 8$:	$A = \{2, 3, \mathbf{4}, 7\}$	$B = \{1, \mathbf{5}, 6, 8\}$
$n = 10$:	$A = \{1, 2, \mathbf{3}, 4, 5\}$	$B = \{6, 7, \mathbf{8}, 9, 10\}$
$n = 12$:	$A = \{1, 2, 3, \mathbf{4}, 6, 8\}$	$B = \{5, 7, \mathbf{9}, 10, 11, 12\}$

We now show that the desired decomposition of M exists for any $n = 2k$, where $k \neq 2$.

If k is odd, then one possible decomposition is given by

$$A = \{1, 2, \dots, k\}, \qquad B = \{k+1, k+2, \dots, 2k\}.$$

The sum of all the elements of A is $\frac{1}{2}k(k+1)$, their arithmetic mean equals $\frac{1}{2}(k+1)$, which is a natural number. Since $1 \leqslant \frac{1}{2}(k+1) \leqslant k$, the arithmetic mean of all the elements of A is an element of A. Similarly, the arithmetic mean $\frac{1}{2}(3k+1)$ of all elements of the subset B is an element of B.

For $k = 4$ the existence of the decomposition is shown in the above table; for even numbers $k \geqslant 6$ a possible decomposition is given by

$$A = \{1, 2, \dots, k-2, k, \tfrac{1}{2}(3k-2)\}, \qquad B = M \setminus A.$$

We have $k < \frac{1}{2}(3k-2) \leqslant 2k$ and $\frac{1}{2}(3k-2)$ is a natural number. The set A thus contains k natural numbers from the set M. The sum of all the elements of A is

$$1 + 2 + \dots + (k-2) + k + \tfrac{1}{2}(3k-2) = \tfrac{1}{2}(k-2)(k-1) + k + \tfrac{1}{2}(3k-2) = \tfrac{1}{2}k(k+2).$$

Their arithmetic mean is $\frac{1}{2}(k+2)$, which is a natural number. Since $1 \leqslant \frac{1}{2}(k+2) \leqslant k-2$, the arithmetic mean of all the elements of A is an element of A. Similarly one shows that the arithmetic mean $\frac{3}{2}k$ of all elements of B is an element of B.

b) Let A, B and C denote the desired subsets of the set M. Since they all have the same number of elements, n must be divisible by 3, hence of the form $n = 3k$, where k is a natural number. The sum s of all elements of M equals $s = \frac{1}{2}3k(3k+1)$. The sum of the three arithmetic means of the elements in the subsets A, B and C, respectively, is thus equal to s/k, that is, $\frac{3}{2}(3k+1)$. By the hypotheses, this sum must be a natural number, thus k must be odd.

On the other hand, for numbers of the form $n = 3k$, where k is odd, a possible decomposition is given by

$$A = \{1, 2, \dots, k\}, \quad B = \{k+1, k+2, \dots, 2k\} \quad \text{and} \quad C = \{2k+1, 2k+2, \dots, 3k\}.$$

Indeed, the sum of all elements in A is $\frac{1}{2}k(k+1)$, hence their arithmetic mean is $\frac{1}{2}(k+1)$, which is a natural number; and since $1 \leqslant \frac{1}{2}(k+1) \leqslant k$, this arithmetic mean is an element of A. Similarly we show that the arithmetic mean $\frac{1}{2}(3k+1)$ of all the elements of B is an element of B, and the arithmetic mean $\frac{1}{2}(5k+1)$ of all the elements of C is an element of C.

Conclusion. In part a), the possible numbers n are all even n different from 4; in part b), all odd n divisible by three.

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5. *In the plane a circle k is given with center S, and a point $A \neq S$. Find the locus of all circumcenters of triangles ABC whose side BC is a diameter of k.*

Solution. Let r be the radius of k. If A lies on k, then S is the circumcenter of any of the triangles ABC, and the sought locus thus reduces to the singleton $\{S\}$. Otherwise we distinguish two cases:

a) Let $|AS| > r$. Consider first the isosceles triangle ABC with basis BC, satisfying the conditions of the problem. The circumcenter O of this triangle is an interior point of the segment AS and at the same time $|AO| = |BO| = |CO|$.

We claim that the sought locus $\mathcal{O}$ is the line p perpendicular to AS and passing through O (Fig. 2).

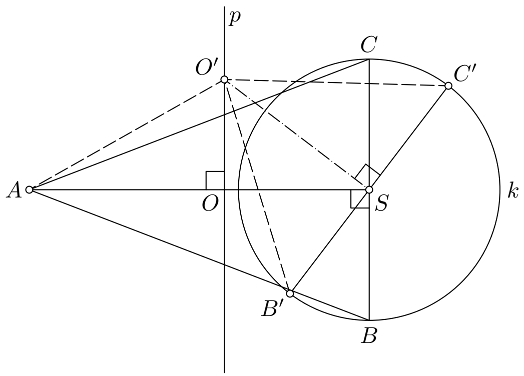

Fig. 2

Consider an arbitrary triangle $AB'C'$, where $B'C'$ is a diameter of k, and denote by O' the intersection of the perpendicular bisector of its side $B'C'$ with the line p, so that $|O'B'| = |O'C'|$ (the point O' lies on the perpendicular bisector of $B'C'$). By the Pythagorean theorem in the right triangle $C'O'S$,

$$|O'B'| = |O'C'| = \sqrt{|O'S|^2 + r^2} = \sqrt{|OO'|^2 + |OS|^2 + r^2}.$$

On the other hand, for the length of the segment $O'A$ we have

$$|O'A| = \sqrt{|AO|^2 + |OO'|^2} = \sqrt{|BO|^2 + |OO'|^2} = \sqrt{|OS|^2 + r^2 + |OO'|^2}.$$

Thus $|O'A| = |O'B'| = |O'C'|$, so the point O' is the circumcenter of the triangle $AB'C'$ and by construction it lies on the line p.

Conversely, for any point O' of the line p it is possible to construct a diameter $B'C'$ of the circle k which is perpendicular to the line $O'S$. By the previous arguments, $|O'A| = |O'B'| = |O'C'|$, so we have found a triangle $AB'C'$ with the required property whose circumcenter is O'.

b) Let $|AS| < r$. This case can be treated in an analogous manner. The center O is now an interior point of the half-line opposite to SA. We arrive at the same result as in the case a).

Conclusion. If A is not a point of k, the sought locus $\mathcal{O}$ is the line p perpendicular to AS which passes through the circumcenter O of the isosceles triangle ABC whose basis BC is the diameter of k perpendicular to AS. If A is a point of k, then $\mathcal{O} = \{S\}$.

Other solution. For the given point $A \notin k$ consider a triangle with the required properties. Denote by l the circumcircle of the triangle ABC (Fig. 3). Since S is the

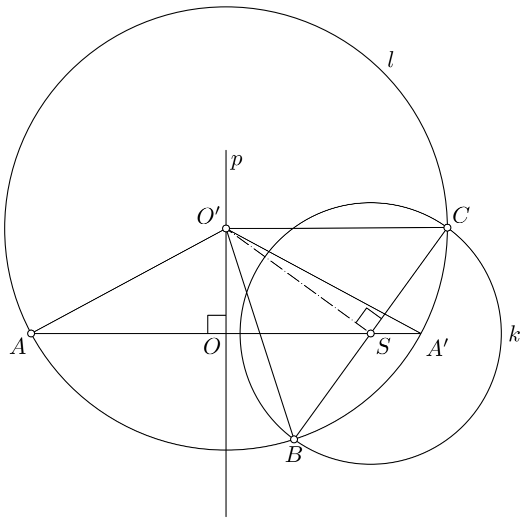

Fig. 3

midpoint of the common chord BC of the circles k and l, the circle l intersects the half-line opposite to SA at an interior point which we denote by A'. For the power $m_l(S)$ of the point S with respect to l we then have

$$m_l(S) = -|BS| \cdot |CS| = -r^2 = -|AS| \cdot |A'S|, \tag{1}$$

where r is the radius of k. It follows that the distance $|A'S|$, hence also the position of the point A' on the half-line opposite to SA, is uniquely determined by the point A. For all triangles ABC satisfying the conditions of the problem, the segment AA' is therefore one and the same. The circumcircles of all the triangles ABC thus have a common chord AA', so their centers lie on the perpendicular bisector p of the segment AA'. In the case of an isosceles triangle ABC with basis BC, the segment AA' is a diameter of l and its center O is the midpoint of AA'. The line p thus passes through this point O and is perpendicular to AS.

Conversely, to each point O' of the line p we find a triangle ABC with the required properties, whose circumcenter coincides with O'. It is enough to construct the diameter BC of the circle k which is perpendicular to the line $O'S$. For given A, A' and S we thus obtain points B and C for which the relation (1) holds. This means that the points A, B, C and A' lie on the same circle l. Since the point O' is the intersection of the chords AA' and BC of this circle, which are not parallel, the point O' is the center of l, and thus is the circumcenter of the triangle ABC.

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6. *Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that for all integers x, y,*

$$f\big(f(x)+y\big) = x + f(y+2006).$$

Solution. Let f be an arbitrary function with the required property. Taking in turn $y = 0$ and $y = 1$, we obtain the equalities

$$f\big(f(x)\big) = x + f(2006), \quad \text{resp.} \quad f\big(f(x)+1\big) = x + f(2007), \tag{1}$$

so upon subtracting

$$f\big(f(x)+1\big) - f\big(f(x)\big) = f(2007) - f(2006).$$

The last relation can be rewritten as

$$f(z+1) - f(z) = f(2007) - f(2006) \tag{2}$$

for all $z \in \mathbb{Z}$ which belong to the range of f. However, this range is all of $\mathbb{Z}$, as is evident from any of the equalities (1).

The validity of (2) for all $z \in \mathbb{Z}$ means that the values of f on $\mathbb{Z}$ form an arithmetic progression (infinite on both sides), so f must be given by a recipe of the form $f(z) = az + b$ for suitable constants $a, b \in \mathbb{R}$. Substituting this into the original equation for f the left-hand and the right-hand sides become

$$f\big(f(x)+y\big) = a\big(f(x)+y\big) + b = a^2x + ay + ab + b,$$
$$x + f(y+2006) = x + a(y+2006) + b = x + ay + 2006a + b.$$

These two expressions are equal for all $x, y \in \mathbb{Z}$ if and only if $a^2 = 1$ and at the same time $2006a = ab$; that is, $a = \pm 1$ and $b = 2006$. The only solutions are thus the two functions

$$f_1(x) = x + 2006 \quad \text{and} \quad f_2(x) = -x + 2006.$$

First Round of the 56th Czech and Slovak Mathematical Olympiad (December 5th, 2006)

1. *Find all real numbers s for which the equation*

$$4x^4 - 20x^3 + sx^2 + 22x - 2 = 0$$

has four distinct real roots and the product of two of these roots is -2.

Solution. Assume that s is a number as above, and denote the four roots of the equation by x_1, x_2, x_3 and x_4 in such a way that

$$x_1 x_2 = -2. \tag{0}$$

From the factorization

$$4x^4 - 20x^3 + sx^2 + 22x - 2 = 4(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

we obtain, upon multiplying out the brackets and comparing the coefficients at like powers of x on both sides, the familiar Vièta's relations

$$x_1 + x_2 + x_3 + x_4 = 5, \tag{1}$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{s}{4}, \tag{2}$$

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -\frac{11}{2}, \tag{3}$$

$$x_1x_2x_3x_4 = -\frac{1}{2}. \tag{4}$$

From the equalities (0) and (4) it follows immediately that

$$x_3x_4 = \frac{1}{4}.$$

Rewriting (3) as

$$(x_1 + x_2)x_3x_4 + (x_3 + x_4)x_1x_2 = -\frac{11}{2}$$

and substituting the known values for x_1x_2 and x_3x_4 we obtain

$$\frac{1}{4}(x_1 + x_2) - 2(x_3 + x_4) = -\frac{11}{2},$$

which together with the equation (1) forms a system of two linear equations for the unknown sums $x_1 + x_2$ and $x_3 + x_4$. An easy calculation shows that its solution is given by

$$x_1 + x_2 = 2 \quad \text{and} \quad x_3 + x_4 = 3.$$

Inserting all this into the equality (2) rewritten in the form

$$x_1x_2 + (x_1 + x_2)(x_3 + x_4) + x_3x_4 = \frac{s}{4},$$

we find that necessarily $s = 17$.

Conversely, from the equalities

$$x_1 + x_2 = 2 \quad \text{and} \quad x_1x_2 = -2$$

it follows that the numbers $x_{1,2}$ are the roots of the quadratic equation

$$x^2 - 2x - 2 = 0, \quad \text{or} \quad x_{1,2} = 1 \pm \sqrt{3}; \tag{5}$$

and from the equalities

$$x_3 + x_4 = 3 \quad \text{and} \quad x_3x_4 = \frac{1}{4}$$

it follows that the numbers $x_{3,4}$ are the roots of the quadratic equation

$$x^2 - 3x + \frac{1}{4} = 0, \quad \text{or} \quad x_{3,4} = \frac{3}{2} \pm \sqrt{2}. \tag{6}$$

We see that $x_{1,2,3,4}$ are indeed four mutually different real numbers which satisfy the system (1)–(4) for the value $s = 17$, hence are the roots of the original equation from the statement of the problem.

There is thus only one such number s, namely $s = 17$.

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2. *Consider the set* $\{1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 80, 160\}$ *and all its three-element subsets. Decide which are more numerous: the three-element subsets for which the product of their elements is greater than 2006, or those for which the product of their elements if less than 2006?*

Solution. The given set is exactly the set of all (natural) divisors of the number $160 = 2^5 \cdot 5$. We can group its elements into pairs in such a way that the product of the numbers in each pair equals 160:

$$1 \cdot 160 = 2 \cdot 80 = 4 \cdot 40 = 5 \cdot 32 = 8 \cdot 20 = 10 \cdot 16.$$

This means that if $A = \{a, b, c\}$ is a triple of mutually distinct divisors of 160, then so is $A' = \{160/a, 160/b, 160/c\}$.

The product abc of the elements of the triple A can be expressed in the form

$$2^k 5^l, \quad \text{where } k \in \{0, 1, 2, \ldots, 14\},\ l \in \{0, 1, 2, 3\}. \tag{1}$$

(The number 160 has only two divisors which are multiples of 2^5, hence in the prime factorization of the number abc there cannot appear the factor 2^{15}.) It is not difficult to see that the largest natural number of the form (1) which is less than 2006 is the number $2000 = 2^4 \cdot 5^3$, and the least natural number which is of the form (1) and is greater than 2006 is $2048 = 2^{11}$ (the number 2006 itself is not of the form (1)). At the same time, $2000 \cdot 2048 = 160^3$.

Consequently, if the product abc of the triple A is less than 2006, then $abc \leqslant 2000$ and the product $160^3/(abc)$ of the corresponding triple A' is at least $160^3/2000 = 2048$. Conversely, if the product abc of the triple A is greater than 2006, then $abc \geqslant 2048$ and the product of the triple A' is at most $160^3/2048 = 2000$. In other words, the three-element subsets whose product of elements is less than 2006 are exactly as numerous as the three-element subsets whose product of elements is greater than 2006.

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3. *A trapezoid $ABCD$ is given, with right angle at the vertex A and with basis AB, in which $|AB| > |CD| \geqslant |DA|$. Denote by S the intersection of the bisectors of its interior angles at the vertices A and B, and by T the intersection of the bisectors of the interior angles at the vertices C and D. Similarly we denote by U, V the intersections of the bisectors of the interior angles at the vertices A and D and at B and C, respectively.*

a) Show that the lines UV and AB are parallel.

b) Show that the intersection E of the half-line DT with the line AB and the points S, T and B are concyclic.

Solution. Being the intersection of the bisectors of the interior angles at the vertices A and D of the given trapezoid, the point U has equal distances from the sides AB and AD as well as from the sides AD and DC. This means that it has the same distance also from the two bases AB, CD of the trapezoid $ABCD$. Similarly the point V has the same distance from both bases. The lines UV and AB must therefore be parallel, which settles part a).

Since the sum of interior angles at the vertices A and D, as well as at the vertices B and C, is $180°$, the sum of the angles adjacent to the side AD of the triangle ADU is equal to $90°$, as is the sum of the angles adjacent to the side BC of the triangle BCV. This means that both these triangles are right (with right angles at the vertices U and V, respectively, Fig. 1). The quadrangle $UTVS$ is therefore chordal (from the hypothesis $|AB| > |CD| \geqslant |DA|$ it follows that the half-lines AU and CV do not

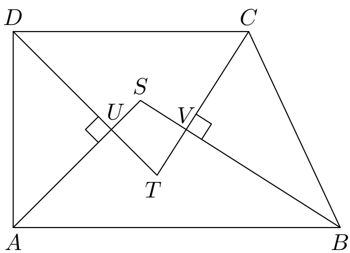

Fig. 1

meet, thus the points S and T lie in the opposite half-planes determined by the line UV and the points U, T, V, and S lie on the circle in the order indicated).

As we already know, the lines UV, AB and CD are parallel, thus $|\angle VUT| = |\angle CDT| = 45°$. From the equality of the arc angles subtended by the chord TV of the chordal quadrangle $UTVS$ it therefore follows that $|\angle VST| = |\angle VUT| = 45°$. This is also the magnitude of the arc angle TSB subtended by the chord TB in the circumcircle of the triangle STB (Fig. 2). It remains to show that on this circle there also lies the point E. This is obvious if $E = T$. Otherwise it is enough to check that the magnitude of the angle TEB is either $180° - 45°$ or $45°$ according as the line BT separates the points S, E or not; however, this follows immediately from the fact that the line DT meets the basis AB at an angle of $45°$ (Fig. 2 and 3). This settles part b).

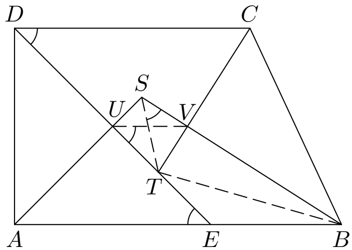

Fig. 2

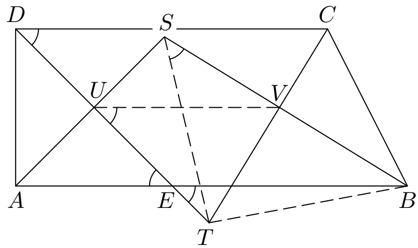

Fig. 3

Second Round of the 56th Czech and Slovak Mathematical Olympiad (January 23rd, 2007)

1. *Find the least possible area of a triangle ABC whose altitudes satisfy the inequalities $h_a \geqslant 3$ cm, $h_b \geqslant 4$ cm and $h_c \geqslant 5$ cm.*

Solution. Denote by a, b, c the lengths of the sides of the triangle ABC. Its altitude h_b satisfies the inequality

$$c \geqslant h_b,$$

since h_b is the length of the shortest segment connecting the vertex B with a point on the line AC. The area S of the triangle ABC therefore satisfies

$$S = \frac{ch_c}{2} \geqslant \frac{h_b h_c}{2} \geqslant 10\,\text{cm}^2.$$

If there exists a triangle ABC satisfying the conditions of the problem whose area is exactly $10\,\text{cm}^2$, then both inequalities $S = \frac{1}{2}ch_c \geqslant \frac{1}{2}h_b h_c \geqslant 10\,\text{cm}^2$ must become equalities. This means that $c = h_b = 4\,\text{cm}$ and at the same time $h_c = 5\,\text{cm}$. The first equality means that the triangle is right, with the right angle at the vertex A. The length of its cathetus AC then satisfies $b = h_c = 5\,\text{cm}$, while the length A of its hypotenuse BC equals $\sqrt{41}\,\text{cm}$. From the formula $S = \frac{1}{2}ah_a$ we obtain for the altitude h_a

$$h_a = \frac{2S}{a} = \frac{20}{\sqrt{41}}\,\text{cm} > 3\,\text{cm}.$$

This means that the right triangle ABC with catheti of lengths $b = 5\,\text{cm}$ and $c = 4\,\text{cm}$ satisfies the conditions of the problem.

The least possible area of the triangle ABC whose altitudes have the requested properties is thus $10\,\text{cm}^2$.

---

2. *Let a, b be real numbers. Prove that if the equation*

$$x^4 - 4x^3 + 4x^2 + ax + b = 0$$

has two distinct real roots such that their sum is equal to their product, then it has no other real roots and $a + b > 0$.

Solution. Assume that the equation

$$x^4 - 4x^3 + 4x^2 + ax + b = 0 \tag{1}$$

has two distinct real roots x_1 and x_2 such that $x_1 + x_2 = x_1 x_2 = p$. Then the polynomial on the left-hand side is divisible by the polynomial $(x - x_1)(x - x_2) = x^2 - px + p$ and has the decomposition

$$x^4 - 4x^3 + 4x^2 + ax + b = (x^2 - px + p)(x^2 + rx + s),$$

where r, s are real numbers. Multiplying out the expression on the right-hand side in the last inequality and comparing the coefficients at the same powers of x on both sides we get

$$-4 = -p + r, \tag{2}$$
$$4 = p + s - pr, \tag{3}$$
$$a = -ps + pr, \tag{4}$$
$$b = ps. \tag{5}$$

From the relation (2) it follows that

$$r = p - 4. \tag{6}$$

Substituting this into (3) we get

$$s = 4 - p + p(p - 4) = (p - 4)(p - 1). \tag{7}$$

Since the quadratic equation $x^2 - px + p = 0$ has two distinct real roots x_1 and x_2, its discriminant is a positive number, so

$$p^2 - 4p > 0. \tag{8}$$

Adding up the equalities (4) and (5) and substituting for r from (6), we arrive at

$$a + b = pr = p(p - 4) = p^2 - 4p > 0,$$

which is what we wanted to prove.

For the discriminant D of the equation

$$x^2 + rx + s = 0$$

it follows from the formulas (6), (7) a (8) that

$$D = r^2 - 4s = (p - 4)^2 - 4(p - 4)(p - 1) = -3p(p - 4) = -3(p^2 - 4p) < 0.$$

The last equation therefore has no real roots. The given equation (1) thus has no other real roots than x_1 and x_2.

3. *Let M be an arbitrary interior point of the hypotenuse AB of a right triangle ABC. Denote by S, S_1, and S_2 the circumcenters of the triangles ABC, AMC, and BMC, respectively.*

a) Show that the points M, C, S_1, S_2 and S lie on a circle.

b) For which position of the point M does this circle have the least radius?

Solution. a) Let α and β be the magnitudes of the interior angles at the vertices A and B of the given right triangle ABC (Fig. 1). From the relation between the central

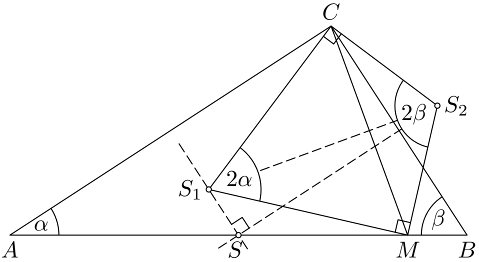

Fig. 1

and the arc angle subtended by the common chord CM in the circumcircles k_1 and k_2 of the triangles AMC and BMC, respectively, we obtain

$$|\angle MS_1C| + |\angle MS_2C| = 2\alpha + 2\beta = 180^\circ.$$

The quadrangle CS_1MS_2 is thus chordal. Since the points M and C are axially symmetric with respect to the perpendicular bisector of the segment CM, and since S_1 and S_2 lie on this bisector, we further have

$$|\angle S_1MS_2| = |\angle S_1CS_2| = 90^\circ.$$

The circumcircle of the quadrangle CS_1MS_2 is thus the Thaletian circle over the diameter S_1S_2. On the other hand, the points S and S_1 lie on the perpendicular bisector of the cathetus AC, and similarly the points S and S_2 lie on the perpendicular bisector of the cathetus BC of the given triangle. Consequently, $|\angle S_1SS_2| = 90^\circ$, and the point S therefore lies also on the Thaletian circle circumscribed to the quadrangle CS_1MS_2. (If $M = S$, then this assertion trivially also holds.) This proves part a).

b) The radius r of the circle (with chord CS) found in part a) clearly satisfies $2r \geqslant |CS|$, with equality taking place if and only if CS is its diameter. Since the circle with diameter CS passes through the midpoints of both catheti AC and BC, the equality $2r = |CS|$ holds if and only if S_1 is the midpoint of AC and S_2 is the midpoint of BC; this clearly corresponds to M being the foot of the altitude from the vertex C onto the hypotenuse AB.

Another solution. a) Denote by P_1 and P_2 the midpoints of the segments AM and BM, respectively (Fig. 2). Since the homothety with center M and coefficient $\frac{1}{2}$

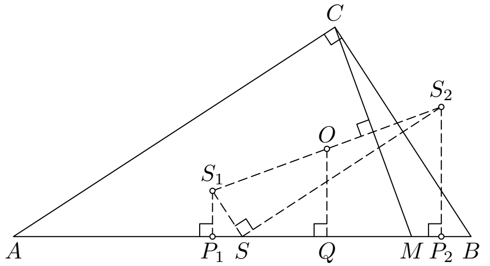

Fig. 2

maps the segment AB onto the segment P_1P_2, it maps the midpoint S of AB into the midpoint Q of P_1P_2, and at the same time as the image of the point S the point Q is the midpoint of the segment MS. The points P_1, P_2 are the orthogonal projections of the points S_1, S_2 onto the hypotenuse AB, so the point Q is the orthogonal projection of the center O of the circle over the diameter S_1S_2. By the Thaletian theorem this circle contains S, since the lines S_1S and S_2S, being the perpendicular bisectors of the two perpendicular catheti AC and BC, are perpendicular. From the symmetry of this circle with respect to the line OQ it then follows that the point M also lies on this circle, whence so does the point C (in view of the symmetry with respect to the line S_1S_2). This proves part a).

b) The segment S_1S_2 and its orthogonal projection P_1P_2 satisfy $|S_1S_2| \geqslant |P_1P_2| = \frac{1}{2}|AB|$. The circumcircle of the quadrangle CS_1MS_2 thus has least diameter $\frac{1}{2}|AB|$, if and only if $S_1S_2 \parallel AB$, which in view of the orthogonality of the segment CM and its perpendicular bisector S_1S_2 takes place if and only if M is the foot of the altitude from the vertex C in the triangle ABC. (The radius r of this circle then is $r = \frac{1}{4}|AB|$.)

Another solution. a) Consider the similarity obtained upon composing the rotation around C by the oriented (right) angle ACB and the homothety with center C and coefficient equal to the ratio $|BC| : |AC|$ (Fig. 3). This similarity maps the points A, B and M into B, B' and M', respectively, where BC is the altitude onto the

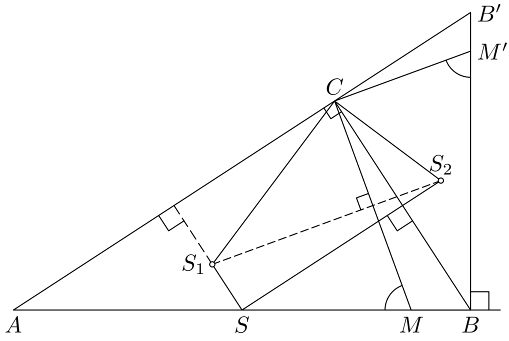

Fig. 3

hypotenuse AB' in the right triangle ABB', and the point M' lies on its cathetus BB'. In view of the congruent angles AMC and $BM'C$ (or also in view of the right angles MCM' and MBM'), we see that the circumcircle of the triangle BMC is at the same time also the circumcircle of the triangle $BM'C$, so its center S_2 is the image of the point S_1 in the above similarity (which maps the triangle AMC exactly onto the triangle $BM'C$). This means that the angle S_1CS_2 is right, hence so is the angle S_1MS_2 (since the line S_1S_2 is the perpendicular bisector of the segment CM). Finally, the angle S_1SS_2 is also right (since its arms lie on the perpendicular bisectors of the two perpendicular catheti AC and BC), which means that all three points C, M, S lie on the Thaletian circle over the diameter S_1S_2.

This proves part a) of the problem. Part b) is solved in the same manner as in the first solution.

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4. *Let natural numbers p, q $(p < q)$ be given. Find the least natural number m with the following property: the sum of all fractions whose denominators (in lowest terms) are equal to m and whose values lie in the open interval (p, q) is at least $56(q^2 - p^2)$.*

Solution. We show that the least m is 113 (independent of p, q). Clearly $m > 1$. For arbitrary natural numbers $c < d$ and $m > 1$, let $S_m(c, d)$ denote the sum of all fractions (in their lowest terms) which lie in the open interval (c, d) and whose denominator is m. Then we have the inequality

$$S_m(c, c+1) \leqslant \left(c + \frac{1}{m}\right) + \left(c + \frac{2}{m}\right) + \cdots + \left(c + \frac{m-1}{m}\right) = (m-1)c + \frac{m-1}{2},$$

with equality taking place if and only if all the numbers $1, 2, \ldots, m-1$ are coprime with m, i.e. if and only if m is a prime.

For any given natural numbers p, q and $m > 1$ we have

$$\begin{aligned} S_m(p, q) &= S_m(p, p+1) + S_m(p+1, p+2) + \cdots + S_m(q-1, q) \\ &\leqslant \left((m-1)p + \frac{m-1}{2}\right) + \left((m-1)(p+1) + \frac{m-1}{2}\right) + \ldots \\ &\quad + \left((m-1)(q-1) + \frac{m-1}{2}\right) = \\ &= (m-1)\frac{(q-p)(p+q-1)}{2} + (m-1)\frac{q-p}{2} = \\ &= (m-1)\frac{q-p}{2}(p+q-1+1) = \frac{(m-1)(q^2-p^2)}{2}, \end{aligned}$$

that is,

$$S_m(p, q) \leqslant \frac{(m-1)(q^2-p^2)}{2}. \tag{9}$$

Moreover, equality takes place in (9) if and only if m is a prime. However, by hypothesis

$$S_m(p, q) \geqslant 56(q^2 - p^2).$$

In view of (9) we see that necessarily $\frac{1}{2}(m-1) \geqslant 56$, i.e. $m \geqslant 113$. As 113 is a prime, the least possible m equals 113.

Final Round of the 56th Czech and Slovak Mathematical Olympiad (March 18–21, 2007)

1. *A chess piece is placed on some square in an $n \times n$ ($n \geqslant 2$) square chessboard. It then makes alternately "straight" and "diagonal" moves. "Straight" means to a square having a common side with the original square. "Diagonal" means to a square which has exactly one point in common with the original square. Find all n for which there exists a sequence of moves, starting by a "diagonal" move from the original square, such that the piece passes through all squares of the chessboard, and through each square exactly once.*

Solution. We first show that the problem has a solution for an arbitrary even n. Indeed, placing the piece e.g. into any of the corners of the chessboard, it is possible to pass through all the squares of the chessboard using the adjacent $2 \times n$ blocks in the manner indicated in Fig. 1 for $n = 8$. Here the sequence of moves corresponds to the sequence of the connecting oriented segments. The argument for a general even n is the same.

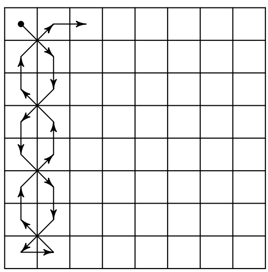

Fig. 1

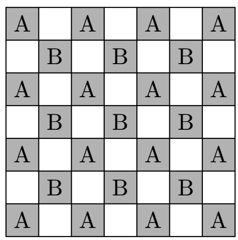

Fig. 2

Now we show that for an odd $n \geqslant 3$ it is not possible to pass through all squares of the chessboard in the manner indicated. Aiming at contradiction, let as assume that for some odd n there exists a sequence of moves on the $n \times n$ chessboard satisfying the conditions of the problem. Let us color all squares of the chessboard in a similar manner as the ordinary 8×8 chessboard in such a way that the squares in the corners are black (as in Fig. 2 for $n = 7$). Further, label all the black squares by the letters A and B in such a way that no two black squares having exactly one point (vertex) in common are labelled by the same letter. If the black squares in the corners are labelled e.g. by the letters A, then the number of A-squares will clearly be greater by n than the number of B-squares.

Let us finally denote the squares of the chessboard which the piece in turn passes through by $1, 2, 3, \ldots, n^2$, and the k-th move of the piece by the notation $k \mapsto k + 1$. If the square 1 is black, then the black squares are exactly those with numbers $1, 2, 5, 6, 9, 10, \ldots$; at the same time, each of the (diagonal) moves $1 \mapsto 2$, $5 \mapsto 6$, $9 \mapsto 10$, ... connects black squares labelled by different letters. It follows that the number of A and B squares differs by at most 1, which is a contradiction. Similarly, if the starting square 1 is white, the black squares are exactly those with numbers $3, 4, 7, 8, 11, 12, \ldots$, connected by the (diagonal) moves $3 \mapsto 4$, $7 \mapsto 8$, $11 \mapsto 12$, ..., and the same contradiction is obtained.

The solution are therefore all even $n \geqslant 2$.

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2. *In a chordal quadrangle $ABCD$ denote by L, M the incenters of the triangles BCA and BCD, respectively. Denote further by R the intersection of the perpendiculars from the points L and M onto the lines AC and BD, respectively. Show that the triangle LMR is isosceles.*

Solution. Let us denote by H the intersection of the bisectors of the interior angles at the vertices A and D in the triangles BCA and BCD (Fig. 3). Then H is the midpoint of the corresponding arc BC of the circumcircle k of the quadrangle $ABCD$ (of the arc not containing the vertices A and D). Denote $\epsilon = |\angle BAH| = |\angle CAH| =$

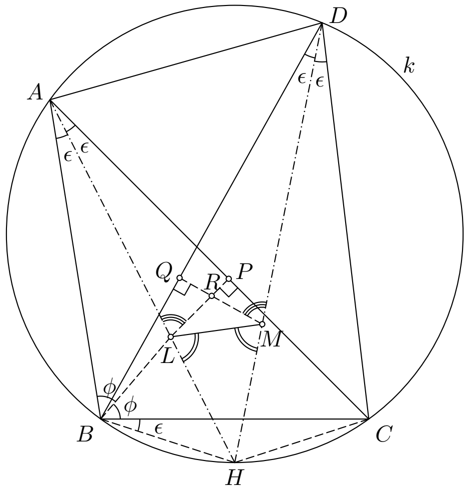

Fig. 3

$|\angle BDH| = |\angle CDH| = |\angle CBH|$ and $\phi = |\angle ABL| = |\angle CBL|$. Then

$$|\angle BLH| = |\angle BAL| + |\angle ABL| = \epsilon + \phi = |\angle LBH|.$$

The triangle HLB is thus isosceles with basis LB, whence $|HB| = |HL|$. Similarly $|HC| = |HM|$. And since $|HB| = |HC|$, we also have $|HL| = |HM|$, so the triangle HML is isosceles and $|\angle HLM| = |\angle HML|$.

Denote further by P the orthogonal projection of the point L onto the line AC, and by Q the orthogonal projection of the point M onto the line BD (the point R in question is thus the intersection of the lines LP and MQ). Since the right triangles APL and DQM have congruent angles at the vertices A and D, the angles PLA and QMD at the vertices L and M are also congruent. From this and from the equality $|\angle HLM| = |\angle HML|$ it therefore follows that $|\angle PLM| = |\angle QML|$. This means that the triangle LMR is isosceles, which is what we wanted to prove.

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3. *Denote by $\mathbb{N}$ the set of all natural numbers and consider all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for any $x, y \in \mathbb{N}$,*

$$f\bigl(xf(y)\bigr) = yf(x).$$

Find the least possible value of $f(2007)$.

Solution. Let f be any function with the given property. We claim first of all that f is injective. Indeed, if $f(y_1) = f(y_2)$, then for all natural x

$$y_1 f(x) = f\bigl(xf(y_1)\bigr) = f\bigl(xf(y_2)\bigr) = y_2 f(x),$$

and as $f(x)$ is a natural number it follows that $y_1 = y_2$.

Taking $x = 1$ in the given equation we get in particular $f\bigl(f(y)\bigr) = yf(1)$, which for $y = 1$ becomes $f\bigl(f(1)\bigr) = f(1)$. As f is injective, this means that

$$f(1) = 1, \tag{1}$$

so that for all natural y

$$f\bigl(f(y)\bigr) = y. \tag{2}$$

The last relation implies, in particular, that the range of the function f is the entire set $\mathbb{N}$. For any natural z we can thus find y such that $y = f(z)$ and at the same time $f(y) = z$; using again the given equation, we therefore get

$$f(xz) = f\bigl(x(f(y)\bigr) = yf(x) = f(z)f(x).$$

An easy induction argument then implies that

$$f(x_1 x_2 \dots x_n) = f(x_1)f(x_2)\dots f(x_n) \tag{3}$$

for any natural numbers n and $x_1, x_2, \dots, x_n$.

Next, we show that the image $f(p)$ of an arbitrary prime p is also a prime. Assume that $f(p) = ab$, where a and b are natural numbers different from 1. By (2) a (3), then

$$p = f\bigl(f(p)\bigr) = f(ab) = f(a)f(b).$$

Since f is injective and $f(1) = 1$, we must have $f(a) > 1$, $f(b) > 1$, contradicting the hypothesis that p is a prime.

Since the decomposition of the number 2007 into prime factors is $2007 = 3^2 \cdot 223$, we thus get by (3)

$$f(2007) = f(3)^2 f(223),$$

where both $f(3)$ and $f(223)$ are primes. If $f(3) = 2$, then $f(2) = 3$ by (2) so the least possible value for $f(223)$ is 5, whence $f(2007) \geqslant 20$. If $f(3) = 3$, then the least possible value of $f(223)$ is 2 and $f(2007) \geqslant 18$. It is easy to see that for any other choice of the values $f(3)$ a $f(223)$ we get $f(2007) \geqslant 18$.

We now show that there exists a function satisfying the conditions of the problem and such that $f(2007) = 18$. Define f in the following manner: For any natural number x, which we write as $x = 2^k 223^m q$, where k and m are nonnegative integers and q is a natural number coprime with 2 and 223, set

$$f(2^k 223^m q) = 2^m 223^k q.$$

Then $f(2007) = f(223 \cdot 3^2) = 2 \cdot 3^2 = 18$. We check that this function f indeed has the required property. Let $x = 2^{k_1} 223^{m_1} q_1$ and $y = 2^{k_2} 223^{m_2} q_2$ be arbitrary natural numbers written in the above form. Then

$$\begin{aligned} f\bigl(xf(y)\bigr) &= f\bigl(2^{k_1} 223^{m_1} q_1 f(2^{k_2} 223^{m_2} q_2)\bigr) = f(2^{k_1+m_2} 223^{m_1+k_2} q_1 q_2) = \\ &= 2^{k_2+m_1} 223^{m_2+k_1} q_1 q_2 \end{aligned}$$

and at the same time

$$yf(x) = 2^{k_2} 223^{m_2} q_2 f(2^{k_1} 223^{m_1} q_1) = 2^{k_2+m_1} 223^{m_2+k_1} q_1 q_2.$$

The least possible value of $f(2007)$ is thus 18.

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4. *The set M contains all natural numbers from 1 to 2007 (inclusive) and has the following property: if $n \in M$, then M contains all the members of the arithmetic progression with first member n and difference $n+1$. Decide whether there must always exist a number m such that M contains all natural numbers greater than m.*

Solution. The answer is in the negative; a counterexample is given by the set

$$M = \mathbb{N} \setminus \bigl\{a\colon a + 1 \text{ is a prime greater than } 2008\bigr\},$$

which clearly contains all natural numbers from 1 to 2007. At the same time, a general member of the arithmetic progression $\bigl(a_n\bigr)_{n=1}^{\infty}$ with first member $a_1 = n \in M$ and difference $d = n + 1$ has the form

$$a_k = a_1 + (k-1)d = n + (k-1)(n+1) = (n+1)k - 1,$$

which implies that $a_k + 1 = (n+1)k$ can never be a prime, for any $k > 1$; thus $a_k \in M$ for all k (no matter whether $a_k \leqslant 2007$ or $a_k \geqslant 2008$). Since there are infinitely many primes, there are also infinitely many numbers not lying in the set M.

5. *An acute triangle ABC is given such that $|AC| \neq |BC|$. In the interior of its sides BC and AC consider the points D and E, respectively, for which $ABDE$ is a chordal quadrangle. Denote by P the intersection of its diagonals AD and BE. Show that if the lines CP and AB are perpendicular, then P is the orthocenter of the triangle ABC.*

Solution. Denote $\phi = |\angle BAD|$ and $\psi = |\angle ABE|$ (Fig. 4). From the equality $|\angle AEB| = |\angle ADB|$ of the angles subtending the chord AB in the chordal quad-

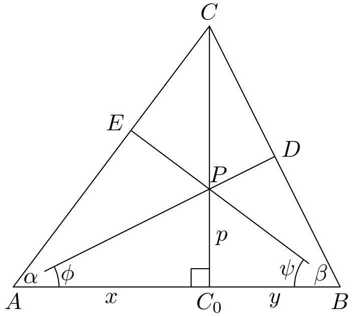

Fig. 4

rangle $ABDE$ we thus obtain (using the standard notation for the angles in the triangle ABC)

$$\alpha + \psi = \beta + \phi. \tag{1}$$

Denote by C_0 the foot of the altitude from the vertex C, by h_c the length of this altitude CC_0, and by x, y and p the lengths of the corresponding segments AC_0, BC_0 and PC_0, respectively (Fig. 4); thus

$$\begin{aligned} \tan\phi &= \frac{p}{x}, \qquad \tan\psi = \frac{p}{y}, \\ \tan\alpha &= \frac{h_c}{x}, \qquad \tan\beta = \frac{h_c}{y}. \end{aligned} \tag{2}$$

If the point P is not the orthocenter (i.e. the angle $\alpha + \psi$ is not right), we can use (1) and write

$$\tan(\alpha + \psi) = \tan(\beta + \phi).$$

Using the well-known addition formula for the tangent, it follows from (2) that (using also the equality $\tan\alpha\tan\psi = \tan\beta\tan\phi$, which likewise follows from (2))

$$\frac{h_c}{x} + \frac{p}{y} = \frac{h_c}{y} + \frac{p}{x}$$

or

$$(p - h_c)(x - y) = 0.$$

Since $p < h_c$ and $x \neq y$ in view of the hypothesis we have made, the last equality cannot hold. Thus $\alpha + \psi = 90°$ and the point P is the orthocenter, which is what we needed to prove.

6. *Find all ordered triples (x, y, z) of mutually distinct real numbers which satisfy the set equation*

$$\{x, y, z\} = \left\{\frac{x-y}{y-z}, \frac{y-z}{z-x}, \frac{z-x}{x-y}\right\}.$$

Solution. If x, y, z are three mutually distinct real numbers, then

$$u = \frac{x-y}{y-z}, \quad v = \frac{y-z}{z-x}, \quad w = \frac{z-x}{x-y} \tag{1}$$

are clearly numbers different from 0 and -1 whose product is equal to 1. This property must therefore be possessed also by the values x, y, z from any such triple. We will thus assume from now on that

$$x, y, z \in \mathbb{R} \setminus \{0, -1\}, \quad x \neq y \neq z \neq x, \quad xyz = 1. \tag{2}$$

Since the given set relation is the same for each of the ordered triples (x, y, z), (z, x, y) and (y, z, x), we will assume in addition to (2) that $x > \max\{y, z\}$, and will distinguish two cases, according as $y > z$ or $z > y$. Let us introduce the following notation for intervals: $I_1 = (0, \infty)$, $I_2 = (-1, 0)$, $I_3 = (-\infty, -1)$.

The case of $x > y > z$. For the fractions (1) we clearly have $u \in I_1$, $v \in I_2$ and $w \in I_3$, so $u > v > w$. The given set equation can thus be fulfilled only when $u = x$, $v = y$ and $w = z$. Upon substituting from (1) and an easy manipulation we arrive at the equations

$$xy + y = yz + z = zx + x, \quad \text{where } x \in I_1,\ y \in I_2,\ z \in I_3. \tag{3}$$

In view of the condition $xyz = 1$ from (2) we can replace the term zx in the equation $xy + y = zx + x$ by $1/y$. This leads to

$$xy + y = \frac{1}{y} + x \Rightarrow x(y-1) = \frac{1-y^2}{y} \Rightarrow x = -\frac{1+y}{y} \Rightarrow y = -\frac{1}{1+x}.$$

(We have used the fact that, as $y \in I_2$, necessarily $y \neq 1$.) From the last formula it follows that the value of the first expression in the system (3) is -1, so from the fact that the second expression equals -1 we obtain

$$z = -\frac{1}{1+y} = -\frac{1}{1 - \dfrac{1}{1+x}} = -\frac{1+x}{x}.$$

But then also the third expression in (3) is equal to -1. Any solution of our problem (in the current case of $x > y > z$) must therefore be of the form

$$(x, y, z) = \left(t, -\frac{1}{1+t}, -\frac{1+t}{t}\right), \tag{4}$$

where $t \in I_1$ is arbitrary (in view of (3), we do not need to worry about checks). From the procedure used it also follows that taking $t \in I_2$ (or $t \in I_3$, respectively) in the formula (4) we get all solutions of our problem satisfying $z > x > y$ ($y > z > x$), so that it is not necessary to list the cyclic permutations of the triples from (4) in our final answer below.

The case of $x > z > y$. Now we have for the fractions (1) $u \in I_3$, $v \in I_1$ and $w \in I_2$, so $v > w > u$, and the set equation in question is satisfied only if $u = y$, $v = x$ and $w = z$. Upon substituting the fractions from (1) we arrive at the system

$$x - y = y(y - z), \quad y - z = x(z - x), \quad z - x = z(x - y). \tag{5}$$

Adding up these three equations yields

$$0 = y(y - z) + x(z - x) + z(x - y) = (y - x)(x + y - 2z),$$

which in view of $x \neq y$ implies that $z = \frac{1}{2}(x + y)$. Substituting this back into (5) we find (taking again into account that $x \neq y$) that the only solution is $x = 1$, $y = -2$ and $z = -\frac{1}{2}$. The same triple also forms the (unique) solution satisfying $y > x > z$, as well as the (unique) solution for which $z > y > x$.

Answer: The solutions are all ordered triples (4), where $t \in \mathbb{R} \setminus \{0, -1\}$, and the three triples (x, y, z) of the form

$$\left(1, -2, -\frac{1}{2}\right), \ \left(-\frac{1}{2}, 1, -2\right), \ \left(-2, -\frac{1}{2}, 1\right).$$

Czech-Slovak-Polish Match
Bílovec, June 25–26, 2007

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1. *Find all polynomials P with real coefficients for which the equality*

$$P(x^2) = P(x) \cdot P(x+2)$$

holds for every real number x.

Solution. The constant polynomial $P(x) = c$ is a solution if and only if $c = c^2$, thus the polynomials $P(x) = 0$ and $P(x) = 1$ are solutions of the problem.

We claim that the only polynomial of a positive degree n which solves the equation is of the form $P(x) = (x-1)^n$. In view of the identity $(x^2-1)^n = (x-1)^n(x+1)^n$, the latter is clearly a solution for any $n \geqslant 1$.

If ax^n ($a \neq 0$) is the leading term of a polynomial $P(x)$ of a positive degree n, then ax^{2n} is the leading term of the polynomial $P(x^2)$ and a^2x^{2n} is the leading term of the polynomial $P(x)P(x+2)$. If P satisfies the given equality, comparing the leading order terms thus gives $a = a^2$, hence $a = 1$. The polynomial P can therefore be written in the form $P(x) = (x-1)^n + Q(x)$, where Q is either identically zero, or is a nonzero polynomial of degree k, where $0 \leqslant k < n$. Comparing the polynomials

$$P(x^2) = (x^2-1)^n + Q(x^2),$$
$$P(x)P(x+2) = [(x-1)^n + Q(x)][(x+1)^n + Q(x+2)]$$

we obtain (upon multiplying out the brackets and cancelling the terms $(x^2-1)^n$ on both sides) the equality

$$Q(x^2) = (x-1)^n Q(x+2) + (x+1)^n Q(x) + Q(x)Q(x+2).$$

The zero polynomial Q clearly satisfies this relation. For a nonzero Q of degree $k < n$, however, $Q(x^2)$ is a polynomial of degree $2k$, while on the right-hand side of the last equation there is a polynomial of degree $n+k$ (whose leading term is $2bx^{n+k}$, if bx^k is the leading order term of the polynomial $Q(x)$). Since $2k < n+k$, this is not possible.

Conclusion. The solutions are the constant polynomials $P(x) = 0$ and $P(x) = 1$ and the polynomial $P(x) = (x-1)^n$ for any natural number n.

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2. *Let $a_1 = a_2 = 1$ and $a_{k+2} = a_{k+1} + a_k$ for any $k \in \mathbb{N}$ (the Fibonacci sequence). Prove that for any natural number m there exists an index k such that the number $a_k^4 - a_k - 2$ is divisible by m.*

Solution. All the congruences and remainder classes below are meant mod m. We obtain the desired congruence relation $a_k^4 - a_k - 2 \equiv 0$ as a consequence of the simpler relation $a_k \equiv -1$.

The sequence of remainder classes of the numbers a_k has the following property: the remainder classes of any two consecutive elements a_k, a_{k+1} determine uniquely the remainder classes of all subsequent elements a_i $(i > k+1)$, as well as of all elements a_i $(i < k)$ preceding them. By the standard argument, based on the fact that the number of ordered pairs of remainder classes is m^2, hence finite, it follows that the sequence of remainder classes of the elements a_i is periodic, starting already from its *first* member. Thus there exists a number $p > 0$ (depending on the given modulus m) such that $a_i \equiv a_{i+p}$ for any index i. Unless $m = 1$ (then the problem is trivial), clearly $p > 1$. Since $a_1 \equiv a_2 \equiv 1$, we also have $a_{p+1} \equiv a_{p+2} \equiv 1$, whence $a_p \equiv 0$ and $a_{p-1} \equiv -1$, so we can take $k = p - 1$ and the proof is finished.

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3. *Let k be the circumcircle of a given convex quadrilateral $ABCD$ with the property that the half-lines DA and CB meet at a point E for which $|CD|^2 = |AD| \cdot |ED|$ holds. Let us denote by F $(F \neq A)$ the point of intersection of the circle k with the perpendicular to ED at A. Prove that the segments AD and CF are congruent if and only if the circumcenter of the triangle ABE lies on ED.*

Solution. Clearly DF is a diameter of k. First we show that under the given conditions the vertex C cannot lie in the half-plane DFA.

If the vertices B, C are points on the subarc DA of the arc DAF (Fig. 1) then the angles DCB and DBA are obtuse, hence $|DC| < |DB| < |DA| < |DE|$, which contradicts to the equality $|CD|^2 = |AD| \cdot |ED|$.

If the vertices B, C are points on the subarc AF of the arc DAF (Fig. 2) the angle BAE is acute and $|\angle DBE| = 180° - |\angle DBC| \leqslant 90°$, so the possible other meeting point B' of the half-line DB with the circumcircle of the triangle AEB lies in the segment DB. Hence $|DC| > |DB| \geqslant |DB'|$. This means that the equality $|CD|^2 = |AD| \cdot |ED|$ cannot hold as $|AD| \cdot |ED| = |DB| \cdot |DB'|$ (which is the power of D with respect to the circumcircle of the triangle AEB).

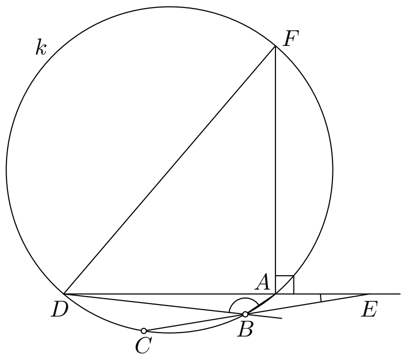

Fig. 1

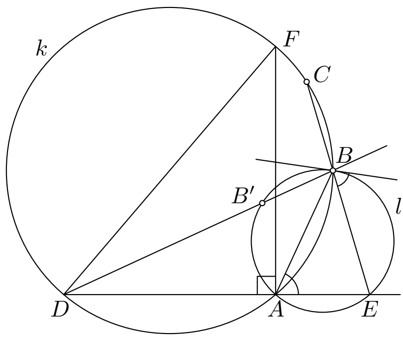

Fig. 2

We have shown that the vertex C of the given quadrangle does not lie in the half-plane FDA, hence $|FC| = |DA|$ if and only if $DAFC$ is a rectangle, i.e. if and only if CA is a diameter of the circle k, which is equivalent to the angle CBA being right, which is in turn equivalent to the triangle AEB being right with the right angle at B, i.e. to the circumcenter of the triangle AEB being the midpoint of AE.

4. *For any real number $p \geqslant 1$ let us consider the set of all real numbers x with*

$$p < x < \left(2 + \sqrt{p + \frac{1}{4}}\right)^2.$$

Prove that from such a given set one can select four mutually different natural numbers a, b, c, d with $ab = cd$.

Solution. The numbers $a = (k-1)k$, $b = (k+1)k$, $c = (k-1)(k+1)$, $d = k^2$ clearly satisfy the equality $ab = cd$ and the inequalities $a < c < d < b$ for any $k > 1$. Let thus k be the least natural number for which $p < a$, i.e. $p < (k-1)k$ (for a given p). We will show that for this k necessarily $b = (k+1)k \leqslant p + 4 + 2\sqrt{4p+1}$, which is evidently a number by $\frac{1}{4}$ smaller than the upper bound of the interval in our problem, so we will be done.

In view of the choice of the number k we have $p \geqslant (k-2)(k-1)$. Solving this quadratic inequality yields the estimate

$$k \leqslant \frac{3}{2} + \sqrt{p + \frac{1}{4}},$$

from which it already follows that

$$\begin{aligned} b = (k+1)k &\leqslant \left(\frac{5}{2} + \sqrt{p + \frac{1}{4}}\right) \cdot \left(\frac{3}{2} + \sqrt{p + \frac{1}{4}}\right) \\ &= \frac{15}{4} + 4\sqrt{p + \frac{1}{4}} + \left(p + \frac{1}{4}\right) = p + 4 + 2\sqrt{4p+1}. \end{aligned}$$

5. *Find for which*

$$n \in \{3\,900, 3\,901, 3\,902, 3\,903, 3\,904, 3\,905, 3\,906, 3\,907, 3\,908, 3\,909\}$$

the set $\{1, 2, 3, \ldots, n\}$ can be partitioned into (disjoint) triples in such a way that one of the three numbers in any triple is the sum of the other two.

Solution. From the possibility of partitioning the set into disjoint triples it follows that $3 \mid n$. In each triple $\{a, b, a+b\}$ the sum of its elements is $2(a+b)$, hence an even number; thus also the sum of all numbers from 1 to n must be even, i.e. the product $n(n+1)$ must be divisible by four. Altogether it therefore follows that the number n has to be of the form either $12k$ or $12k+3$; from the given set of numbers, this is satisfied only for $n = 3\,900$ and $n = 3\,903$.

In the next paragraph we describe a construction how to produce, starting from a decomposition satisfying the given condition for some $n = k$, a decomposition of the same kind for $n = 4k$ and $n = 4k+3$. This guarantees that the required decompositions for $n = 3\,900$ and $n = 3\,903$ indeed exist, in view of the decreasing sequence

$$3\,900 \to 975 \to 243 \to 60 \to 15 \to 3$$

(instead of 3 900 one can start also with 3 903) and the trivial decomposition for $n = 3$ (from which we in turn construct the decompositions for $n = 15$, $n = 60$ etc. up to $n = 3\,900$ or $n = 3\,903$).

From a decomposition of the set $\{1, 2, \ldots, k\}$ satisfying the given conditions we first produce a similar decomposition for the set of the first k even numbers $\{2, 4, \ldots, 2k\}$ (simply by multiplying all the numbers in the triples by two). In the case of $n = 4k$ we partition the remaining numbers

$$\{1, 3, 5, \ldots, 2k-1, 2k+1, 2k+2, \ldots, 4k-1, 4k\}$$

into the k triples $\{2j-1, 3k-j+1, 3k+j\}$, where $j = 1, 2, \ldots, k$. They are shown in the columns of the table below.

$$\begin{pmatrix} 1 & 3 & 5 & \ldots & 2k-3 & 2k-1 \\ 3k & 3k-1 & 3k-2 & \ldots & 2k+2 & 2k+1 \\ 3k+1 & 3k+2 & 3k+3 & \ldots & 4k-1 & 4k \end{pmatrix}$$

In the case of $n = 4k+3$ we partition the remaining numbers

$$\{1, 3, 5, \ldots, 2k-1, 2k+1, 2k+2, \ldots, 4k+2, 4k+3\}$$

into the $k+1$ triples $\{2j-1, 3k+3-j, 3k+j+2\}$, where $j = 1, 2, \ldots, k+1$; these are again shown in the columns of the table below.

$$\begin{pmatrix} 1 & 3 & 5 & \ldots & 2k-1 & 2k+1 \\ 3k+2 & 3k+1 & 3k & \ldots & 2k+3 & 2k+2 \\ 3k+3 & 3k+4 & 3k+5 & \ldots & 4k+2 & 4k+3 \end{pmatrix}$$

This completes the proof of the fact that the solution of the given problem are the numbers $n = 3\,900$ and $n = 3\,903$.

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6. *Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that*

$$|\angle PAB| + |\angle PDC| \leqslant 90^\circ \quad \textit{and} \quad |\angle PBA| + |\angle PCD| \leqslant 90^\circ.$$

Prove that $|AB| + |CD| \geqslant |BC| + |AD|$.

Solution. If P is a common point of the given circles, the familiar properties of the angles subtending a chord at a point on a given circle and at its center imply that P is also the point of tangency if and only if (Fig. 3)

$$|\angle ADP| + |\angle BCP| = |\angle APB|. \tag{1}$$

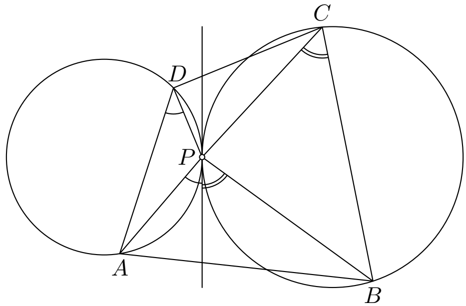

Fig. 3

Consider now the circumcircles of the triangles ABP and CDP and assume for the moment that they meet also at another point Q $(Q \neq P)$.

Since the point A lies outside the circle BCP, we have $|\angle BCP|+|\angle BAP| < 180°$. Therefore the point C lies outside the circle ABP. Analogously, D also lies outside that circle. It follows that P and Q lie on the same arc CD of the circle CDP.

Analogously, the points P and Q lie on the same arc AB of the circle ABP. Thus the point Q lies either inside the angle BPC or inside the angle APD. Without loss of generality assume that Q lies inside the angle BPC (Fig. 4). Then

$$|\angle AQD| = |\angle PQA| + |\angle PQD| = |\angle PBA| + |\angle PCD| \leqslant 90°, \tag{2}$$

under the condition of the problem.

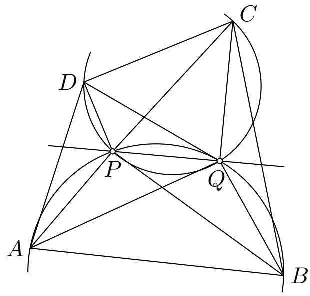

Fig. 4

In the chordal quadrilaterals $APQB$ and $DPQC$, it follows from the hypothesis of the problem that the angles at the vertices A and D are acute. Thus the corresponding opposite angles at the vertex Q are obtuse. This implies that Q lies not only inside the angle BPC but in fact inside the triangle BPC, hence also inside the quadrilateral $ABCD$.

From the properties of the angles in the two chordal quadrilaterals just mentioned it thus follows that

$$|\angle BQC| = |\angle PAB| + |\angle PDC|,$$

so by the hypothesis of the problem

$$|\angle BQC| \leqslant 90^\circ. \tag{3}$$

Moreover, since $|\angle PCQ| = |\angle PDQ|$, we get by (1)

$$\begin{aligned}|\angle ADQ| + |\angle BCQ| &= |\angle ADP| + |\angle PDQ| + |\angle BCP| - |\angle PCQ| \\ &= |\angle ADP| + |\angle BCP|.\end{aligned}$$

The last sum is equal to $|\angle APB|$, according to the observation (1) applied to $T = P$. Since also $|\angle APB| = |\angle AQB|$, we obtain

$$|\angle ADQ| + |\angle BCQ| = |\angle AQB|.$$

This however means, as we have seen in the beginning, that the circles BCQ and DAQ are externally tangent at Q, contradicting our initial assumption that $P \neq Q$. Thus it has to be the case that the circumcircles of the two triangles ABP and CDP have only the single point P in common, for which, by the inequalities (2) and (3), it is further true that the angles APD and BPC are not obtuse.

Consider now the half-discs with diameters BC and DA constructed inwardly to the quadrilateral $ABCD$. Since the angles APD and BPC are not obtuse, these two half-discs lie entirely inside the circles BQC and AQD; and since these two circles are externally tangent, the two half-discs cannot have any other point than P in common. Denoting by M and N the midpoints of the sides BC and DA, respectively, it thus follows that $|MN| \geqslant \frac{1}{2}(|BC| + |DA|)$.

On the other hand, since $\mathbf{MN} = \frac{1}{2}(\mathbf{BA} + \mathbf{CD})$, we have $|MN| \leqslant \frac{1}{2}(|AB| + |CD|)$. Thus indeed $|AB| + |CD| \geqslant |BC| + |DA|$, as claimed.