2010

59th Czech and Slovak Mathematical Olympiad

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First Round of the 59th Czech and Slovak Mathematical Olympiad
Problems for the take-home part
(October 2009)

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1. *Find all real solutions of the system*

$$\begin{aligned}\sqrt{x^2-y}&=z-1,\\ \sqrt{y^2-z}&=x-1,\\ \sqrt{z^2-x}&=y-1.\end{aligned}$$

Solution. The left hand sides of the equations are square roots, i.e. non-negative, thus right hand sides are non-negative as well and $z \geqslant 1$, $x \geqslant 1$, $y \geqslant 1$.

We square the equations to get

$$x^2-y=(z-1)^2, \quad y^2-z=(x-1)^2, \quad z^2-x=(y-1)^2,$$

and sum up and simplify:

$$\begin{aligned}(x^2-y)+(y^2-z)+(z^2-x)&=(z-1)^2+(x-1)^2+(y-1)^2,\\ (x^2+y^2+z^2)-(x+y+z)&=(z^2+x^2+y^2)-2(z+x+y)+3,\\ x+y+z&=3.\end{aligned}$$

On the other hand, summing up the inequalities $x \geqslant 1$, $y \geqslant 1$ a $z \geqslant 1$ we obtain $x+y+z \geqslant 3$, therefore $x=y=z=1$ is the only possibility. The triple $(x,y,z)=(1,1,1)$ is indeed a solution.

Conclusion: The problem has the unique solution $(x,y,z)=(1,1,1)$.

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2. *Let $ABCD$ be a rhombus and let a tangent of its incircle cut the sides BC and CD, and denote R, S the intersections of the tangent with the lines AB, BC respectively. Prove that the value of the product $|BR|\cdot|DS|$ is independent of the choice of the tangent.*

Solution. Let U, V, W, T be the points of tangency of the incircle with the sides AB, BC, DA, and with the tangent respectively (Fig. 1). Further let X be the intersection point of the tangent and the side BC, and let $a=|AB|=|AD|$, $b=|BU|=|BV|=|DW|$ be fixed quantities, while $r=|BR|$ and $s=|DS|$ are the variables dependent on the choice of the tangent. We will show that the product $|BR|\cdot|DS|$ $(=r\cdot s)$ equals to $a\cdot b$.

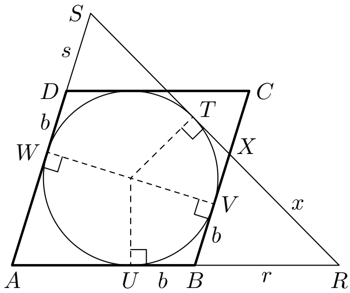

Fig. 1

The point R is the center of similitude of the triangles ARS and BRX. Moreover the incircle (of the triangle ARS) is simultaneously being the excircle of the triangle BRX tangent to the side BX. According to the well-known fact, the center of BX is the symmetry center of the points of tangency of the incircle and of the excircle with BX. This means that the ratio $|SW| : |AR|$ in the triangle ARS corresponds to the ratio $|BV| : |BR|$ in the triangle BRX, that is

$$\frac{b+s}{a+r} = \frac{b}{r}, \quad \text{and} \quad r \cdot s = a \cdot b.$$

which completes the proof.

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3. *There are numbers 1, 2, ..., 33 written on a blackboard. In one step we choose two numbers on the blackboard such that one of them divides the other one, we erase the two numbers and write their integer quotient instead. We proceed in this manner until no number on the blackboard divides another one. What is the least possible amount of numbers left on the blackboard?*

Solution. In the process, apparently only numbers from the set $M = \{1, 2, \ldots, 33\}$ can be on the blackboard. Prime numbers 17, 19, 23, 29 a 31 are going to stay on the blackboard, in one copy each, because they have no other divisor than the number 1 and the set M does not contain any other of their multiples.

We explain now, why there must always be two other numbers on the blackboard. the product of all numbers is

$$S = 33! = 2^{31} \cdot 3^{15} \cdot 5^{7} \cdot 7^{4} \cdot 11^{3} \cdot 13^{2} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \tag{1}$$

at the beginning.

In each step we choose a pair (x, y) with $x \mid y$, that is numbers of the form $x = a$, $y = ka$ and we replace them with the number $y/x = k$. the product S of all the numbers on the blackboard changes to the new value S/a^2. It is evident, that the parity of the exponent of each prime number divisor in the prime number factorization of S is preserved. Especially the four primes 2, 3, 5, and 11 divide the product S throughout the process. Since $2 \cdot 3 \cdot 5 \cdot 11 > 33$ it follows that on the table there always must be at least two numbers which product is divisible by $2 \cdot 3 \cdot 5 \cdot 11$. On the whole, there must always be at least seven numbers on the blackboard.

The sequence of steps

$$32,16 \to 2, \quad 30,15 \to 2, \quad 28,14 \to 2, \quad 26,13 \to 2, \quad 24,12 \to 2, \quad 22,11 \to 2,$$
$$27,9 \to 3, \quad 21,7 \to 3, \quad 18,6 \to 3, \quad 25,5 \to 5, \quad 20,4 \to 5, \quad 8,2 \to 4,$$
$$5,5 \to 1, \quad 4,2 \to 2, \quad 3,3 \to 1, \quad 3,3 \to 1, \quad 2,2 \to 1, \quad 2,2 \to 1, \quad 2,2 \to 1.$$

leaves numbers 17, 19, 23, 29, 31, 10, 33, and seven numbers 1 on the blackboard. The 1s can be eliminated in seven further steps, which leaves just seven numbers on the blackboard.

Conclusion: the least amount of numbers which can be left on the blackboard is seven.

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4. *In an acute-angled triangle with pairwaise different sides let O, V, and S be the circumcenter, the orthocentre, and the incenter respectively. Prove that the perpendicular bisector of the segment OV meets S if and only if one of the inner angles of the triangle ABC is* $60°$.

Solution. First, we show that in any acute-angled triangle ABC holds

$$\gamma = 60° \quad \Longleftrightarrow \quad |CO| = |CV|, \tag{1}$$

where γ is the angle by C. Let us consider triangles CVA_0 and COB_1, where A_0 is the foot of the altitute through A, and B_1 is the center of AC (see Fig. 2). From the triangle ACA_0 we get

$$\gamma = 60° \iff |CA_0| = \frac{|AC|}{2} \iff |CA_0| = |CB_1|.$$

The last equality is between the legs of two right triangles with the equal angles VCA_0 and OCB_1 with the value $90° - \beta$ (to see this for the angle VCA_0 look at the right triangle BCC_0; as for the angle OCB_1, observe that B_10C is the half of the central angle corresponding to the circumferential angle $ABC = \beta$ in the circumcircle of ABC). Thus the equality of the legs CA_0 and CB_1 is equivalent to the equality of the hypotenuses CO and CV, which proves (1).

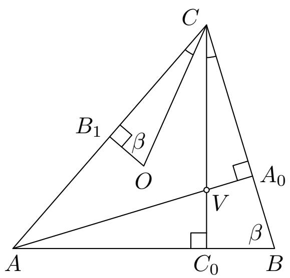

Fig. 2

The angles VCA_0 and OCB_1 are equal, hence CS (recall that S is the incenter) is in any acute-angled triangle ABC not only the bisector of the angle ACB but the bisector of the angle OCV as well. If $\gamma = 60°$ then $|CO| = |CV|$ and the line CS is a perpendicular bisector of the base OV of the equilateral triangle OVC (the points O and V are two different points, since ABC has different sides), in other words the perpendicular bisector of OV meets S. The same is true if $\alpha = 60°$ or $\beta = 60°$.

On the contrary, suppose that the perpendicular bisector of OV meets S and none of the inner angles is $60°$. According (1), $|AO| \neq |AV|$, $|BO| \neq |BV|$, and $|CO| \neq |CV|$. Consider the triangle OVC again, the bisector CS of the inner angle OCV is different from the perpendicular bisector of the base OV, that is its intersection S lies on the circumcircle of OCV according to the well known fact. Equivalently, C lies on the circumcircle of OVS. But the same arguments put points A and B on the circle as well, which means the circumcircle of ABC passes through its center O, which is impossible. This completes the proof.

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5. *They were n fishermen and they caught r_0 fishes together and put them in the fish tank. They come one by one to take their part of the catch. Everyone thinks he is the first one at the tank and after letting one fish free he takes exactly $1/n$ of the current number of fishes in the tank. Determine the smallest possible r_0 (depending on $n \geqslant 2$) if everyone takes home at least one fish.*

Solution. Let us denote r_k the number of fishes in the tank after the k-th fisherman took his fishes, $k = 1, 2, \ldots, n$.

These numbers are determined by r_0 and the reccurence formula

$$r_{k+1} = \frac{n-1}{n}(r_k - 1) \quad (k = 0, 1, \ldots, n-1).$$

For convenience, we rewrite the formulas as

$$r_{k+1} = q \cdot r_k + d, \quad \text{kde} \quad q = \frac{n-1}{n} \quad \text{a} \quad d = \frac{1-n}{n}. \tag{1}$$

First, we derive the explicit expression of the sequence given by the linear recurrence formula $r_{k+1} = q \cdot r_k + d$ ($q, d = \text{const.}$). For $q = 1$ it is just an arithmetic progression, for $q \neq 1$ we have

$$\begin{aligned}
r_1 &= qr_0 + d,\\
r_2 &= qr_1 + d = q(qr_0 + d) + d = q^2 r_0 + (q+1)d,\\
r_3 &= qr_2 + d = q(q^2 r_0 + (q+1)d) + d = q^3 r_0 + (q^2 + q + 1)d,\\
r_4 &= qr_3 + d = q(q^3 r_0 + (q^2 + q + 1)d) + d = q^4 r_0 + (q^3 + q^2 + q + 1)d,\\
&\ \vdots
\end{aligned}$$

and we get the expression

$$r_k = q^k r_0 + (q^{k-1} + q^{k-2} + \cdots + q + 1)d = q^k\Big(r_0 + \frac{d}{q-1}\Big) - \frac{d}{q-1}.$$

Thus

$$r_k = q^k r_0 + \frac{(q^k - 1)d}{q-1} = q^k\Big(r_0 + \frac{d}{q-1}\Big) - \frac{d}{q-1}.$$

In our case

$$\frac{d}{q-1} = \frac{(1-n)/n}{(n-1)/n-1} = n-1,$$

and

$$r_k = \frac{(n-1)^k(r_0+n-1)}{n^k} - n + 1 \quad (k = 0, 1, 2 \ldots, n).$$

Since the numbers $(n-1)^k$ and n^k are relatively prime, the values r_k are integer if and only if the number $r_0 + n - 1$ is divisible by n^k, for all $k = 0, 1, 2, \ldots, n$, that is by n^n. Hence there exists an integer j such that $r_0 + n - 1 = j \cdot n^n$, which is $r_0 = j \cdot n^n - n + 1$, and we can express r_k as

$$r_k = j \cdot (n-1)^k \cdot n^{n-k} - n + 1 \quad (k = 0, 1, 2 \ldots, n). \tag{2}$$

Now we are searching for the least integer $j \geqslant 1$ such that all r_k are positive. Since these numbers form a decreasing sequence, the smallest one is $r_n = j \cdot (n-1)^n - n + 1$, which is for $n \geqslant 3$ positive already for $j = 1$ ($r_0 = n^n - n + 1$), while for $n = 2$ the least j is equal to 2 ($r_0 = 2 \cdot 2^2 - 1 = 7$).

Conclusion: The least possible r_0 is 7 for $n = 2$ and $r_0 = n^n - n + 1$ for $n \geqslant 3$.

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6. *For a given prime p, determine the number of tuples (a, b, c), consisting of numbers from the set $\{1, 2, 3, \ldots, 2p^2\}$, which satisfy*

$$\frac{[a,c]+[b,c]}{a+b} = \frac{p^2+1}{p^2+2} \cdot c,$$

where $[x, y]$ denotes the least common multiple of x and y.

Solution. Let us transform the equation using the well-known relation $(x, y) \cdot [x, y] = x \cdot y$, where (x, y) is the greatest common divisor of x and y. If we denote $u = (a, c)$, and $v = (b, c)$ then the LHS reads

$$\frac{[a,c]+[b,c]}{a+b} = \frac{ac/u + bc/v}{a+b} = \left(\frac{a}{u} + \frac{b}{v}\right) \cdot \frac{c}{a+b}.$$

Thus the original equation is equivalent to (after the multiplication with $(a+b)/c$)

$$\frac{a}{u} + \frac{b}{v} = \frac{p^2+1}{p^2+2} \cdot (a+b). \tag{1}$$

Since $p^2 > 0$ we have the following estimates for the RHS

$$\frac{1}{2} < \frac{p^2+1}{p^2+2} < 1,$$

consequently

$$\frac{a+b}{2} < \frac{a}{u} + \frac{b}{v} < a + b. \tag{2}$$

The left inequality implies u and v cannot both be greater than 1, as inequalities $u \geqslant 2$ and $v \geqslant 2$ imply

$$\frac{a}{u} + \frac{b}{v} \leqslant \frac{a+b}{2}.$$

On the other hand, the right inequality excludes $u = v = 1$, thus exactly one of u and v is 1. Due to symmetry it is sufficient to deal with $u = 1$ and $v \geqslant 2$.

Since $v = (b, c)$, we have $b/v = b_1$, with positive integer b_1. We substitute $u = 1$ and $b = b_1 v$ into (1) and solve it with respect to a:

$$\begin{aligned} a + b_1 &= \frac{p^2+1}{p^2+2} \cdot (a + b_1 v), \\ (p^2+2)(a+b_1) &= (p^2+1)(a+b_1 v), \\ a &= b_1((p^2+1)v - p^2 - 2). \end{aligned} \tag{3}$$

If $v \geqslant 3$ then from the last equation

$$a \geqslant (p^2+1)v - p^2 - 2 \geqslant 3(p^2+1) - p^2 - 2 = 2p^2 + 1,$$

which is a contradiction, as $a \in \{1, 2, 3, \ldots, 2p^2\}$.

Hence we have $v = 2$ and (3) becomes

$$a = b_1(2(p^2+1) - p^2 - 2) = p^2 b_1,$$

which is easy to solve with a in the given domain. Since $a \leqslant 2p^2$ we have $b_1 \leqslant 2$ and since $u = (a, c) = 1$ and $v = (b, c) = 2$ we have c is even and relatively prime with a. Further $a = p^2 b_1$ implies $b_1 = 1$ and p is an *odd* prime. Hence $a = p^2 b_1 = p^2$ and $b = b_1 v = 1 \cdot 2 = 2$, which means *$c$ is even and not a multiple of p.* There are $p^2 - p$ such numbers in $\{1, 2, 3, \ldots, 2p^2\}$ and the tuples $(a, b, c) = (p^2, 2, c)$ are the solutions of the given equation. Because of the symmetry there is the same number of the solutions of the form $(a, b, c) = (p^2, 2, c)$.

Conclusion: There are no tuples if $p = 2$. If p is an odd prime, there are $2(p^2 - p)$ tuples.

First Round of the 59th Czech and Slovak Mathematical Olympiad (December 1st, 2009)

1. *Find all real solutions of the system*

$$\sqrt{x-y^2}=z-1,$$
$$\sqrt{y-z^2}=x-1,$$
$$\sqrt{z-x^2}=y-1.$$

Solution. The square roots and their arguments have to be positive, therefore $x,y,z \geqslant 1$, $x \geqslant y^2$, $y \geqslant z^2$, and $z \geqslant x^2$. The last three inequalities imply $x \geqslant y^2 \geqslant y \geqslant z^2 \geqslant z \geqslant x^2$, and since $x \geqslant 1$, the inequality $x \geqslant x^2$ forces $x=1$, and all the values in the given chain of inequalities are equal, especially $x=y=z=1$, which is indeed a solution.

Conclusion. There is a unique solution $x=y=z=1$.

2. *Find all possible values of the quotient*

$$\frac{r+\rho}{a+b},$$

where r and ρ are respectively the radii of the circumcircle and incircle of the right triangle with legs a and b.

Solution. The distances between the vertices and points of tangency of the incircle (denoted according to the figure) of a triangle ABC are

$$|AU|=|AV|=\frac{b+c-a}{2},\quad |BV|=|BT|=\frac{a+c-b}{2},\quad |CT|=|CU|=\frac{a+b-c}{2},$$

which is easy to obtain from

$$|AV|+|BV|=c,\quad |AU|+|CU|=b,\quad |BT|+|CT|=a.$$

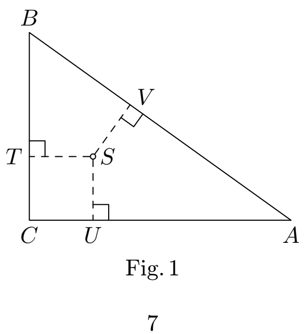

Fig. 1

The points C, T, U, and S (the center of the incircle) form a deltoid, actually a square, if the angle ACB is right. The side of the square is $\rho = |SU| = |CU|$. This gives

$$\rho = \frac{a+b-c}{2};$$

moreover according to the Thales theorem $r = \frac{1}{2}c$ and we get

$$r + \rho = \frac{c}{2} + \frac{a+b-c}{2} = \frac{a+b}{2}.$$

Conclusion. There is only one possible value of the quotient $(r+\rho)/(a+b)$ in any right triangle, namely $\frac{1}{2}$.

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3. *There are numbers* $1, 2, \ldots, 33$ *written on the blackboard. In one step we choose a group of numbers on the blackboard (at least two) such that their product is a square, we erase them and write the square root of their product instead. We proceed until no group can be chosen. What is the least amount of numbers left on the blackboard?*

Solution. The product of all the numbers written on the blackboard is

$$S = 2^{31} \cdot 3^{15} \cdot 5^7 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31.$$

Apparently, the numbers 17, 19, 23, 29, and 31 can never be erased and can never be a part of any change. In any step, there is always left at least one other number, which gives in total $5+1=6$ numbers. Six numbers on the blackboard are achievable indeed: because of the odd exponents of primes 2, 3, 5, and 11 in S we allocate the set $A = \{2, 9, 11, 22, 25\}$ and we put all the other numbers with exception of 17, 19, 23, 29 a 31 into set

$$B = \{3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 18, 20, 21, 24, 26, 27, 28, 30, 32, 33\}.$$

In the first step we choose the group A and replace it by the number

$$n = \sqrt{2 \cdot 9 \cdot 11 \cdot 22 \cdot 25} = \sqrt{2^2 \cdot 3^2 \cdot 5^2 \cdot 11^2} = 2 \cdot 3 \cdot 5 \cdot 11.$$

Because the product of all the numbers from B is $2^{31-2} \cdot 3^{15-2} \cdot 5^{7-2} \cdot 7^4 \cdot 11^{3-2} \cdot 13^2 = 2^{29} \cdot 3^{13} \cdot 5^5 \cdot 7^4 \cdot 11 \cdot 13^2$, in the second step we choose n together with all the numbers from B and replace it by

$$\sqrt{(2 \cdot 3 \cdot 5 \cdot 11) \cdot (2^{29} \cdot 3^{13} \cdot 5^5 \cdot 7^4 \cdot 11 \cdot 13^2)} = 2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13,$$

which leaves just six numbers on the blackboard.

Conclusion. The sought least amount is six.

Second Round of the 59th Czech and Slovak Mathematical Olympiad (January 20th, 2010)

1. *Prove that the equation $x^2 + p|x| = qx - 1$ with real parameters p, q has four real solutions if and only if $p + |q| + 2 < 0$.*

Solution. Evidently, 0 is not a solution of the equation for any p, q. Thus the solutions of the equation are the positive solutions of

$$x^2 + px = qx - 1 \quad \text{i.e.} \quad x^2 + (p - q)x + 1 = 0, \tag{1}$$

together with the negative solutions of

$$x^2 - px = qx - 1 \quad \text{i.e.} \quad x^2 - (p + q)x + 1 = 0. \tag{2}$$

Since any quadratic equation has at most two solutions, the original equation has four solutions if and only if the equation (1) has two positive roots and the equation (2) has two negative roots. Now we find out, when this happens.

First, both of the discriminants $(p - q)^2 - 4$ and $(p + q)^2 - 4$ of the equations (1) and (2) have to be positive, that is

$$(p - q)^2 > 4 \quad \text{and} \quad (p + q)^2 > 4. \tag{3}$$

Further the smaller root of the equation (1) has to be positive and the bigger root of (2) has to be negative, that is

$$\frac{q - p - \sqrt{(p - q)^2 - 4}}{2} > 0 \quad \text{and} \quad \frac{p + q + \sqrt{(p + q)^2 - 4}}{2} < 0. \tag{4}$$

We rewrite the first inequality:

$$q - p > \sqrt{(p - q)^2 - 4}, \tag{5}$$

and we get $q - p > 0$, which together with (3) means $q - p > 2$. Then the inequality (5) is satisfied as well, because

$$q - p = \sqrt{(p - q)^2} > \sqrt{(p - q)^2 - 4}.$$

Thus we have proved that (1) has two different positive solutions if and only if $q - p > 2$, which is the first condition in

$$p - q + 2 < 0, \quad p + q + 2 < 0. \tag{6}$$

Analogously we rewrite the second condition in (4) as

$$\sqrt{(p+q)^2-4} < -(p+q) \qquad \text{(which implies } p+q<0\text{)}$$

which for negative $p+q$ gives $p+q<-2$, The inequalities (6) are thus equivalent to the statement in question.

But is is easy to verify

$$(p-q+2<0 \ \wedge\ p+q+2<0) \ \Leftrightarrow\ p+|q|+2<0.$$

Since $|q|=\max\{-q,q\}$ implies

$$p+|q|+2=\max\{p-q+2,p+q+2\},$$

and the maximum of two real numbers is negative if and only if both of them are negative.

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2. *Let $ABCD$ be a parallelogram with the obtuse angle ABC. We choose a point P on the diagonal AC and in the halfplane BDC such that $|\angle BPD|=|\angle ABC|$. Prove that the line CD is tangent to the circumcircle of the triangle BCP, if and only if $AB=BD$.*

Solution. The line BD separates points A and P, and

$$|\angle BAD|+|\angle BPD|=|\angle BAD|+|\angle ABC|=180^\circ,$$

which shows that the quadrangle $ABPD$ is cyclic (Fig. 1), thus

$$|\angle DBP|=|\angle DAP|=|\angle DAC|=|\angle ACB|=|\angle BCP|.$$

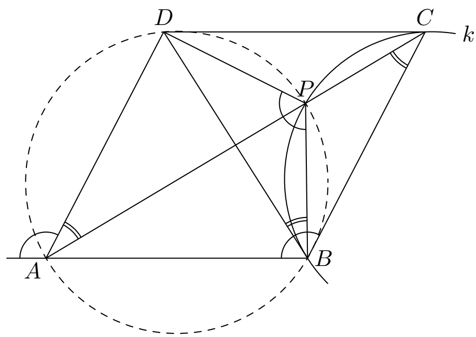

Fig. 1

Since the line BP separates the points C and D, we can use the tangent-chord theorem for the circumcircle k of the triangle BCP to conclude that BD is tangent to k (with the touch point in B). The proof of the problem is easy now:

(i) If the line CD is tangent to k, the symmetry of CD and BD implies $|CD|=|BD|$, that is $|AB|=|BD|$.

(ii) In the opposite direction, if $|AB|=|BD|$, that is $|CD|=|BD|$, the point D lies on the perpendicular axis of the chord BC of the circle k, and CD and BC are symmetric with respect to this axis. Then not only BD is tangent to k but its symmetry image CD as well.

3. *Find all positive integers m, n, such that n divides $2m-1$ and m divides $2n-1$.*

Solution. We seek the pairs of positive integers m, n, such that

$$2m-1=kn \quad \text{a} \quad 2n-1=lm. \tag{1}$$

for some positive integers k, l.

Considering k, l as parameters, we can eliminate the variable n, or we can eliminate the variable m to get:

$$(4-kl)m=k+2, \quad \text{or} \quad (4-kl)n=l+2; \tag{2}$$

Since the RHS of both equation is positive, the LHS must be positive as well and we get $4-kl>0$, which is $kl<4$. We go through cases $kl=1$, $kl=2$ a $kl=3$ one by one.

If $kl=1$, then $k=l=1$ and the equations (2) read $3m=3$ and $3n=3$, which is $m=n=1$.

Since m and n have to be odd, it cannot be $kl=2$.

If $kl=3$ then $\{k,l\}=\{1,3\}$, and we get $m=5$ and $n=3$, or $m=3$ and $n=5$.

Conclusion. The solution are pairs $(1,1)$, $(3,5)$, and $(5,3)$.

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4. *In a triangle ABC, let O be the incenter, P the excenter opposite A, and D the intersection of the bisector of the angle A and the side BC.*
Prove

$$\frac{2}{|AD|}=\frac{1}{|AO|}+\frac{1}{|AP|}.$$

Solution. We will use the known formulas

$$\rho=\frac{2S}{a+b+c} \quad \text{and} \quad \rho_a=\frac{2S}{b+c-a}.$$

for the sides a, b, and c, ρ the incircle radius, ρ_a the radius of the excircle opposite A, and S, the area of ABC.

Since O and P lie on the axis of the angle A, ρ and ρ_a are legs of the right triangles with hypotenuses AO and AP respectively, opposite the angle $\frac{1}{2}\alpha$ (Fig. 2), thus $\rho=|AO|\sin\frac{1}{2}\alpha$, and $\rho_a=|AP|\sin\frac{1}{2}\alpha$. We can now express the right hand side of the given equality as

$$\frac{1}{|AO|}+\frac{1}{|AP|}=\frac{\sin\frac{1}{2}\alpha}{\rho}+\frac{\sin\frac{1}{2}\alpha}{\rho_a}=$$
$$=\frac{\big((a+b+c)+(b+c-a)\big)\sin\frac{1}{2}\alpha}{2S}=\frac{(b+c)\sin\frac{1}{2}\alpha}{S}.$$

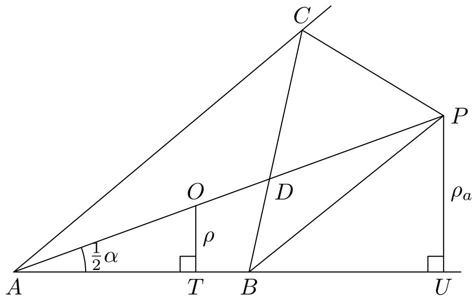

Fig. 2

On the other hand, the area S is the sum of the areas of ABD and ACD, and we can calculate these using the lengths of their sides from A and the angle by A, which is in both triangles $\frac{1}{2}\alpha$.

$$S = S_{ABD} + S_{ACD} = \frac{c|AD| \sin \frac{1}{2}\alpha}{2} + \frac{b|AD| \sin \frac{1}{2}\alpha}{2} = \frac{(b+c)|AD| \sin \frac{1}{2}\alpha}{2}.$$

And we get

$$\frac{2}{|AD|} = \frac{(b+c) \sin \frac{1}{2}\alpha}{S}.$$

Thus the LHS and the RHS have the same value which finishes the proof.

Let us remark that using $S = \frac{1}{2}bc \sin \alpha = bc \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha$ we can extend the equality in question as

$$\frac{2}{|AD|} = \frac{1}{|AO|} + \frac{1}{|AP|} = \frac{b+c}{bc \cos \frac{1}{2}\alpha}.$$

Final Round of the 59th Czech and Slovak Mathematical Olympiad (March 22–23, 2010)

1. *Find all positive integers a a b, such that*

$$4^a + 4a^2 + 4 = b^2.$$

Solution. The equation implies, that b^2 is even and greater than 4^a, which means that b is even and greater than 2^a, that is $b \geqslant 2^a + 2$, and

$$4^a + 4a^2 + 4 = b^2 \geqslant (2^a + 2)^2 = 4^a + 4 \cdot 2^a + 4.$$

which gives $a^2 \geqslant 2^a$, and consequently $a \leqslant 4$.

Substituting the possible values $a = 1, 2, 3, 4$ to the original equation we find out solutions $(a, b) = (2, 6)$ and $(a, b) = (4, 18)$.

2. *Nineteen shots hit the circular target with the radius 12 cm. Prove, that there are two hits separated by a distance smaller than 7 cm.*

Solution. Let $r = 4\sqrt{3}\,\text{cm}$ and we divide the whole target (of radius $r\sqrt{3}$) into 18 disjoint parts. The first six parts are circular sectors with central angle 60°, which together form the circle of radius r in the center of the target. Then we divide the remaining annulus into twelve equal annular sectors with central angle 30° (Fig. 1).

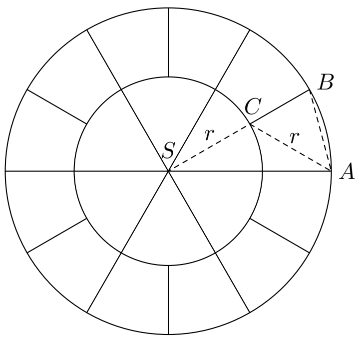

Fig. 1

According to the picture, let us denote as S the center of the target, and A, B, C the vertices of the annular segments.

Since the circles bordering the annular segments have radii r and $r\sqrt{3}$, and since $\cos 30^\circ = \frac{1}{2}\sqrt{3}$, the triangle SAC is isosceles, and thus $|AC| = r$; moreover AC is the longest side in the triangle ABC, which has inner angles 45°, 75°, and 60°. That's why the maximal distance of two points lying in the same annular sector is r, which is the maximal distance of two points in one of the six sectors of the radii r as well. The pigeonhole principle gives us, that there are two hits in the same part, that is their distance is at most $r = 4\sqrt{3} < 7$.

Remark. Let us consider the statement: *From N points inside the circle of radius $r\sqrt{3}$, some two of them are separated by a distance at most r.*

If we want to prove such a statement by comparing the sum of areas of N equal circles of radius r with the area of the circle of radius $r\left(1+2\sqrt{3}\right)$, we succeed, if and only if

$$N \cdot \frac{\pi r^2}{4} > \frac{\pi r^2 \left(1+2\sqrt{3}\right)^2}{4} \quad \text{or} \quad N > 13 + 4\sqrt{3} \doteq 19{,}9.$$

In the given problem, there is even stronger estimate of the distance of two points, namely $r_1 = r \cdot \dfrac{7}{4\sqrt{3}}$, and the similar condition gives

$$N \cdot \frac{\pi r_1^2}{4} > \frac{\pi \left(r_1 + 2r\sqrt{3}\right)^2}{4}, \quad \text{after substitution} \quad N > \left(1 + \frac{24}{7}\right)^2 \doteq 19{,}6.$$

That is we cannot prove the given problem with this “naive” approach.

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3. *A wizard kidnapped 31 members of the political party A, 28 members of the party B, 23 members of the party C, and 19 members of the party D and kept them separately in single cells on his castle and made them to do some work. Each day after the work the party members could walk on the courtyard and talk to each other. But when three members of different parties started to talk to each other, the wizard re-registered them into the fourth party. (More than 3 kidnapped party members never talked to each other)*

a) *Is it possible, that after some time all the kidnapped people were members of one political party? Which one?*

b) *Find all the quadruples of positive integers with the sum 101 which admit, considered as numbers of kidnapped party members, that after some time they become, with the “help” of the wizard, the members of a single party.*

Solution. a) Let a, b, c, and d be the numbers of the kidnapped members of parties A, B, C, and D. The initial quadruple $(a, b, c, d) = (31, 28, 23, 19)$ is according to the parity of the numbers of type (o, e, o, o), where o stands for odd, and e for an even integer. Since the parity of all the numbers a, b, c, and d changes in any re-registration, the quadruple of type (o, e, o, o) becomes the quadruple of type (e, o, e, e), this one then again changes the type to (o, e, o, o) and so forth. If we get after some time the quadruple consisting of 101 and three zeros, the quadruple has to be of type (e, o, e, e), that is all the kidnapped will be the members of the party B, and the following table of changes shows, that this is really possible:

a:	31	30	29	28	27	26	25	24	23	22	...	0
b:	28	27	26	25	24	23	26	29	32	35	...	101
c:	23	22	25	24	27	26	25	24	23	22	...	0
d:	19	22	21	24	23	26	25	24	23	22	...	0

b) We show, that the sought quadruples (a,b,c,d) are those, which contain three numbers which are the same modulo 4.

The equality $a+b+c+d = 101$ implies that three of a, b, c, d have the same parity and the fourth one the opposite one. Because of the symmetry we can suppose that we have

$$a \equiv b \equiv c \not\equiv d \pmod 2$$

for the original quadruple. In any re-registration three of the numbers a, b, c, and d increase by 3, and the fourth one decreases by 1, that is the differences $a-b$, $a-c$, $b-c$ remain the same modulo 4. If in the end $a=b=c=0$, than the said differences have to be divisible by 4 already at the beginning, that is

$$a \equiv b \equiv c \pmod 4. \tag{1}$$

for the original a, b, and c.

Let us show that this condition is sufficient as well. Obviously it is enough to show, that the original quadruple (a,b,c,d) satisfying the condition (1) can be after some steps (re-registrations) changed to the quadruple of the form (e,e,e,f). Namely, after that it suffices to repeat the step $(e,e,e,f) \to (e-1,e-1,e-1,f+3)$.

Now, let (a,b,c,d) be a quadruple with the sum 101, which fulfills the condition (1), and let us further suppose that $a=b=c$ does not hold true. We show, which steps to use to increase the value of d (by 1 or 2). Since $d \leqslant 101$ always, those steps can be applied only finitely many times until $a=b=c$.

It is sufficient to describe the steps in case $a \geqslant b \geqslant c$ and $a > c$, which is $a-c \geqslant 4$ because of (1).[1] Two changes

$$(a,b,c,d) \to (a-1,b-1,c+3,d-1) \to (a-2,b-2,c+2,d+2),$$

enlarge d by 2. These two steps cannot be taken only if $b=1$, but then (1) and $b \geqslant c$ imply $c=1$. A quadruple $(a,1,1,d)$, where $a \geqslant 5$ and $d \geqslant 2$ (it cannot be $d=1$, because of the parity reason) can be transformed by the following three steps:

$$(a,1,1,d) \to (a-1,4,0,d-1) \to (a-2,3,3,d-2) \to (a-3,2,2,d+1),$$

which increase d by 1.

This finishes the proof of the statement about satisfactory quadruples.

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4. *We are given a circle k with a chord AC, which is not a diameter. On its tangent through A we choose a point $X \neq A$ and let D be the intersection point of k with the interior of the segment XC (if it exists). We complete the triangle ACD into the trapezoid $ABCD$ inscribed into k. Determine the set of intersection points of lines BC and AD of such trapezoids.*

Solution. Let us further consider only such trapezoids $ABCD$, where $AB \parallel CD$ (there is no intersection of lines BC and AD if $BC \parallel AD$).

[1] Let us point out that we do not exclude $c=0$. This is vital, because we end up with such a quadruple if the wizard uses two steps described further for $b=2$.

Let O be a center of circle k, E the intersection of its tangents through A and C (Fig. 2). According to Thales, A and C lie on the circle τ with the diameter OE, moreover they are symmetric with respect to that diameter. We denote by ϕ the angles by A and C in the triangle ACE, and further let k_1 and k_2 be the interiors of longer and shorter arc AC of the circle k respectively.

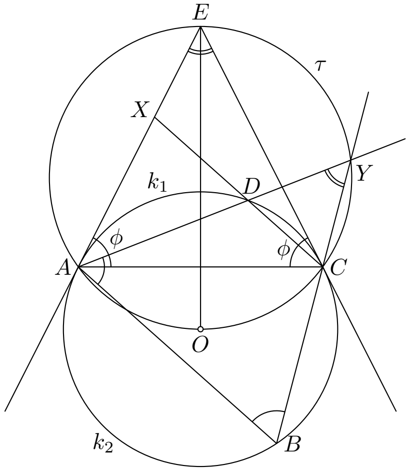

Fig. 2

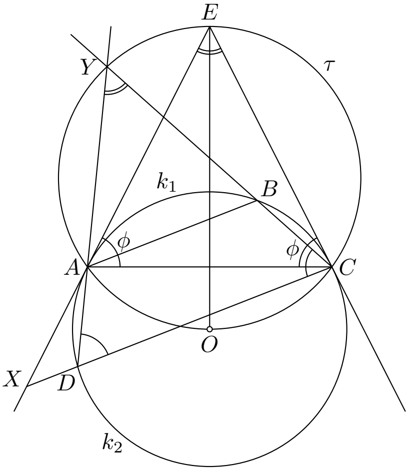

Fig. 3

a) Let X be any point on the tangent AE, $X \neq A$. The circle k meets the segment XC in an inner point D, if and only if X lies in the interior of AE, or in the interior of the half-line opposite to the half-line AE. We deal with these cases (Fig. 2 and Fig. 3) separately.

In the first case $D \in k_1$ and $B \in k_2$, and ϕ and acute angle ABC are the same (ABC is inscribed angle into k_2 subtending the arc AC, ϕ is the tangent-chord angle corresponding to the chord AC). Moreover the angle BAD is the same as well, because every inscribed trapezoid is isosceles. Thus Y, the intersection of half-lines BC and AD, lies in the half-plane ACE. Triangles ABY and ACE are isosceles, thus the angles AYC and AEC are the same $(\pi - 2\phi)$, and Y is on the arc AEC of τ. More precisely, Y is inside the the shorter arc CE, because AD lies in the angle CAE.

The second case is analogous, we write it down briefly: $D \in k_2$, $B \in k_1$, $|\angle ADC| = \phi = |\angle BCD|$, the intersection Y of the half-lines CB and DA lies in the half-plane ACE, and since $|\angle AYC| = |\angle AEC|$, Y lies on τ, inside the shorter arc AE.

b) Now we show, that any point Y inside the shorter arcs CE and AE of τ is the intersection of BC and AD of some of the thought trapezoids $ABCD$. As before, there are two cases.

If Y lies inside the arc CE, obviously there are points $D \in k_1$ and $B \in k_2$ such that A, D, and Y lie on a line respectively, as well as B, C, and Y. Now $D \in k_1$

and therefore there is an intersection X of the half-line CD with the interior of the segment AE (D corresponds to X according to the construction from the statement of the problem). Now we explain why $AB \parallel CD$. Since O and Y lie on different arcs AC of τ and $|AO| = |CO|$, the half-line YO is the bisector of the angle AYC, and $A(D)Y$ a $B(C)Y$ are symmetric along the line YO, which is a line of symmetry of the circle k, because it goes through its center. Thats why the intersections of lines $A(D)Y$ a $B(C)Y$ with k, have to be symmetric along YO as well, that is A and B, and further D and C are symmetric. Thus AB and CD are perpendicular to YO, and consequently parallel.

If Y is inside the arc AE, we construct $D \in k_2$ and $B \in k_1$ in such a way, to have D, A, and Y on a line respectively, and C, B, and Y on a line respectively as well. The half-line CD meets the half-line AE in the desired point X (since $D \neq A$, we have $X \neq A$), if $|\angle AEC| + |\angle ECD| < \pi$. This is really the case. Since $|\angle ECD| = \pi - |\angle CAD| = |\angle CAY|$ according to the inscribed angle theorem, and since $|\angle AEC| = |\angle AYC|$, the sum $|\angle AEC| + |\angle ECD|$ is equal to the sum of two inner angles of the triangle ACY. Further $D(A)Y$ and $C(B)Y$ are symmetric along OY, the bisector of AYC, and we have $AB \parallel CD$ again.

Conclusion. The set in question is the union of interiors of shorter arcs CE and AE of the circle τ.

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5. *There are numbers 1, 2, ... , 33 written on a blackboard. In one step we choose two numbers on the blackboard such that their product is a square, we erase the two numbers and write their square root instead. We proceed in this manner until no product of any two numbers on the blackboard is a square. Prove, that there are at least 16 numbers left on the blackboard.*

Solution. In one step we replace numbers a and b with one positive integer $\sqrt{ab}$. Since $a \leqslant b$ gives $a \leqslant \sqrt{ab} \leqslant b$, it is obvious that only numbers from the set $M = \{1, 2, \ldots, 33\}$ can be on the table. If a is prime or a product of different primes, those primes have to be present in the prime factorization of $\sqrt{ab}$, which is $\sqrt{ab} = ka$ and $b = k^2 a$ for some positive integer k. If $k = 1$, there must be multiple occurrence of a on the table. If $k \geqslant 2$, that is $b = k^2 a \geqslant 4a$, we have $4a \leqslant 33$, and $4a \in M$. Summing up, on the table there are always numbers which divide exactly one number from M, and also those $a \in M$, which are the product of different primes and satisfy $4a > 33$ which is $a \geqslant 9$. Thus we have showed, there are 15 numbers, which have to be on the blackboard:

$$10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33.$$

We show, that beside them there is always at least one more number from the set $S = \{6, 12, 18, 24\}$ there (there are all of them there at the beginning). If we choose a and b in one step, where $a \in S$, and replace them with $n = \sqrt{ab}$, then n is a multiple of 6, and since $a \leqslant 24$ and $b \leqslant 33$ we have $n \leqslant \sqrt{24 \cdot 33} = 6\sqrt{22} < 30$, that is $n \in S$. All together there are always 15 numbers (mentioned above) on the blackboard and one number from S, which finishes the prove.

Remark. Those 16 numbers are really achievable, for example with the following

17 steps (the erased numbers are grey, the new one is at the end of the next line):

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33;
1,2,3,4,5,6,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,29,30,31,32,33,14;
1,2,3,4,5,6,8,9,10,11,12,13,15,16,17,18,19,20,21,22,23,24,25,26,27,29,30,31,32,33,14;
1,2,3,4,6,8,9,10,11,12,13,15,16,17,18,19,21,22,23,24,25,26,27,29,30,31,32,33,14,10;
1,2,3,6,8,9,10,11,12,13,15,16,17,18,19,21,22,23,24,26,27,29,30,31,32,33,14,10,10;
1,2,3,6,8,9,11,12,13,15,16,17,18,19,21,22,23,24,26,27,29,30,31,32,33,14,10,10;
1,2,3,6,8,9,11,12,13,15,16,17,18,19,21,22,23,24,26,27,29,30,31,32,33,14,10;
1,2,3,6,8,9,11,13,15,16,17,18,19,21,22,23,24,26,29,30,31,32,33,14,10,18;
1,2,3,8,9,11,13,15,16,17,18,19,21,22,23,26,29,30,31,32,33,14,10,18,12;
1,2,3,8,9,11,13,15,16,17,19,21,22,23,26,29,30,31,32,33,14,10,12,18;
1,3,8,9,11,13,15,16,17,19,21,22,23,26,29,30,31,32,33,14,10,12,6;
1,3,9,11,13,15,16,17,19,21,22,23,26,29,30,31,33,14,10,12,6,16;
1,3,9,11,13,15,17,19,21,22,23,26,29,30,31,33,14,10,12,6,16;
3,9,11,13,15,17,19,21,22,23,26,29,30,31,33,14,10,12,6,4;
9,11,13,15,17,19,21,22,23,26,29,30,31,33,14,10,6,4,6;
11,13,15,17,19,21,22,23,26,29,30,31,33,14,10,6,6,6;
11,13,15,17,19,21,22,23,26,29,30,31,33,14,10,6,6;
11,13,15,17,19,21,22,23,26,29,30,31,33,14,10,6.

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6. *Find the minimum value of*

$$\frac{a+b+c}{2}-\frac{[a,b]+[b,c]+[c,a]}{a+b+c},$$

where a, b, c are integers grater than 1 and $[x,y]$ denotes the least common multiple of x and y.

Solution. Because of the symmetry it suffices to work with (a,b,c), where $a \geqslant b \geqslant c$. For the "least" of them, that is for $(2,2,2)$, $(3,2,2)$, $(3,3,2)$, $(3,3,3)$, and $(4,2,2)$ the expression in question has values 2, $3/2$, $17/8$, $7/2$, and $11/4$ respectively.

We show that $3/2$ is the minimal value, namely we show that (a,b,c), which satisfy $a+b+c \geqslant 9$, also fulfill

$$\frac{a+b+c}{2}-\frac{[a,b]+[b,c]+[c,a]}{a+b+c} \geqslant \frac{3}{2},$$

We change the inequality equivalently:

$$(a+b+c)^2-2([a,b]+[b,c]+[c,a]) \geqslant 3(a+b+c),$$
$$a^2+b^2+c^2+2(ab-[a,b])+2(bc-[b,c])+2(ca-[c,a]) \geqslant 3(a+b+c).$$

Since $xy \geqslant [x,y]$ for any x, y, we neglect the non-negative multiples on the left hand side and we prove the (stronger) inequality

$$a^2+b^2+c^2 \geqslant 3(a+b+c). \tag{1}$$

The assumption $a+b+c \geqslant 9$ and Cauchy inequality $3(a^2+b^2+c^2) \geqslant (a+b+c)^2$ gives

$$a^2+b^2+c^2 \geqslant \frac{(a+b+c)^2}{3} = 3(a+b+c)\cdot\frac{a+b+c}{9} \geqslant 3(a+b+c),$$

which concludes the prove.