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62nd Czech and Slovak Mathematical Olympiad

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First Round of the 62nd Czech and Slovak Mathematical Olympiad
Problems for the take-home part
(October 2012)

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1. *Find all pairs of primes p, q for which there exists a positive integer a such that*

$$\frac{pq}{p+q} = \frac{a^2+1}{a+1}.$$

(Ján Mazák, Róbert Tóth)

Solution. First, we will deal with the case when the wanted primes p and q are distinct. Then, the numbers pq and $p+q$ are relatively prime: the product pq is divisible by two primes only (namely p and q), while the sum $p+q$ is divisible by neither of these primes.

We will look for a positive integer r which can be a common divisor of both $a+1$ and a^2+1. If $r \mid a+1$ and, at the same time, $r \mid a^2+1$, then $r \mid (a+1)(a-1)$ and also $r \mid (a^2+1)-(a^2-1) = 2$, so r can only be one of the numbers 1 and 2. Thus the fraction $\dfrac{a^2+1}{a+1}$ either is in lowest terms, or will be in lowest terms when reduced by two, depending on whether the integer a is even, or odd.

If a is even, we must have

$$pq = a^2+1 \quad \text{and} \quad p+q = a+1.$$

The numbers p, q are thus the roots of the quadratic equation $x^2-(a+1)x+a^2+1=0$, whose discriminant

$$(a+1)^2 - 4(a^2+1) = -3a^2+2a-3 = -2a^2-(a-1)^2-2$$

is apparently negative, so the equation has no solution in the real numbers.

If a is odd, we must have (taking into account the reduction by two)

$$2pq = a^2+1 \quad \text{and} \quad 2(p+q) = a+1.$$

The numbers p, q are thus the roots of the quadratic equation $2x^2-(a+1)x+a^2+1=0$, whose discriminant is negative as well.

Therefore, there is no pair of distinct primes p, q satisfying the conditions.

It remains to analyze the case of $p=q$. Then,

$$\frac{p\cdot q}{p+q} = \frac{p\cdot p}{p+p} = \frac{p}{2},$$

so we must have

$$p = \frac{2(a^2+1)}{a+1} = 2a-2+\frac{4}{a+1};$$

this is an integer if and only if $a+1 \mid 4$, i. e. $a \in \{1,3\}$, so $p=2$ or $p=5$.

To summarize, there are exactly two pairs of primes satisfying the conditions, namely $p=q=2$ and $p=q=5$.

2. *Two circles $k_1(S_1, r_1)$ and $k_2(S_2, r_2)$ are externally tangent and both lie in a square $ABCD$ with side length a so that k_1 touches the sides AD and CD, while k_2 touches the sides BC and CD. Prove that the area of at least one of the triangles AS_1S_2, BS_1S_2 is no more than $\frac{3}{16}\,a^2$.* (Tomáš Jurík)

Solution. The line segments AS_2 and BS_1 lie on the diagonals of the given square, so they are perpendicular to each other and intersect at the center P of the square. We have

$$|DS_1| = r_1 \cdot \sqrt{2}, \quad |BS_1| = (a - r_1)\sqrt{2}, \quad |PS_1| = \left(\frac{a}{2} - r_1\right)\sqrt{2},$$
$$|CS_2| = r_2 \cdot \sqrt{2}, \quad |AS_2| = (a - r_2)\sqrt{2}, \quad |PS_2| = \left(\frac{a}{2} - r_2\right)\sqrt{2}.$$

Therefore, the area of the triangle AS_1S_2 is

$$S_{AS_1S_2} = \frac{1}{2}|AS_2| \cdot |PS_1| = (a - r_2)\left(\frac{a}{2} - r_1\right),$$

while the area of the triangle BS_1S_2 is

$$S_{BS_1S_2} = \frac{1}{2}|BS_1| \cdot |PS_2| = (a - r_1)\left(\frac{a}{2} - r_2\right).$$

The sum of these areas is

$$S = (a - r_2)\left(\frac{a}{2} - r_1\right) + (a - r_1)\left(\frac{a}{2} - r_2\right) = a^2 - \frac{3}{2}a(r_1 + r_2) + 2r_1r_2.$$

Let K denote the point at which the circle k_1 touches the side AD, H and L denote the points at which k_2 touches the sides CD and BC, respectively, and M be the intersection point of the lines KS_1 and HS_2 (Fig. 1).

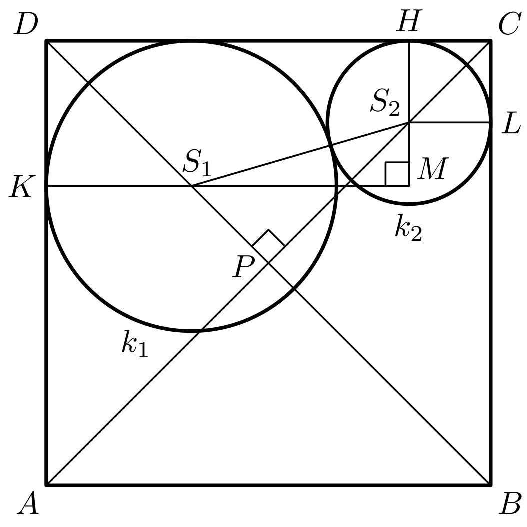

Fig. 1

By the Pythagoras' theorem for the triangle S_1MS_2, we have

$$(a - r_1 - r_2)^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2.$$

Hence we obtain

$$\begin{aligned}(a-r_1-r_2)^2 &= 4r_1r_2,\\ a-r_1-r_2 &= 2\sqrt{r_1r_2},\\ a = r_1+r_2+2\sqrt{r_1r_2} &= \left(\sqrt{r_1}+\sqrt{r_2}\right)^2 \geqslant 4\sqrt{r_1r_2},\end{aligned}$$

i. e.

$$r_1r_2 \leqslant \frac{a^2}{16}.$$

The length of the segment DC cannot be greater than the length of the polygonal chain KS_1S_2L, so

$$2r_1+2r_2 \geqslant a.$$

(This follows from the equality $a = r_1+r_2+2\sqrt{r_1r_2}$ as well since $2\sqrt{r_1r_2} \leqslant r_1+r_2$, by the AM-GM inequality.) Therefore,

$$S = a^2 - \frac{3}{2}a(r_1+r_2) + 2r_1r_2 \leqslant a^2 - \frac{3}{4}a^2 + \frac{1}{8}a^2 = \frac{3}{8}a^2.$$

This means that at least one of the areas $S_{AS_1S_2}$, $S_{BS_1S_2}$ is at most $\frac{3}{16}a^2$.

Other Solution. We can set $a=1$. The difference of the areas of the triangles AS_1S_2 and BS_1S_2 is (according to the expression from the original solution)

$$S_{AS_1S_2} - S_{BS_1S_2} = (1-r_2)\left(\frac{1}{2}-r_1\right) - (1-r_1)\left(\frac{1}{2}-r_2\right) = \frac{1}{2}(r_2-r_1).$$

Without loss of generality, we can assume that $r_1 \geqslant r_2$. Then $S_{AS_1S_2} \leqslant S_{BS_1S_2}$. Now, let us calculate the area of the triangle AS_1S_2. By the Pythagoras' theorem, we have $(1-r_1-r_2)^2+(r_1-r_2)^2 = (r_1+r_2)^2$, hence $\sqrt{r_1}+\sqrt{r_2} = 1$, so $r_2 = \left(1-\sqrt{r_1}\right)^2$. Let us denote $x = \sqrt{r_1}$. It follows from the inequalities $r_1+r_2 \geqslant \frac{1}{2}$ and $r_1 \geqslant r_2$ that $r_1 \geqslant \frac{1}{4}$, and, on the other hand, we have $r_1 \leqslant \frac{1}{2}$ since the circle k_1 lies in the square $ABCD$. Hence it follows that $\frac{1}{2} \leqslant x \leqslant \sqrt{\frac{1}{2}}$. The area of the triangle AS_1S_2 is

$$\begin{aligned}S_{AS_1S_2} =&(1-r_2)\left(\frac{1}{2}-r_1\right) = \frac{1}{2} - r_1 - \frac{r_2}{2} + r_1r_2 =\\ =&\frac{1}{2} - x^2 - \frac{1}{2}(1-x)^2 + x^2(1-x)^2 = x^4 - 2x^3 - \frac{1}{2}x^2 + x;\\ S_{AS_1S_2} - \frac{3}{16} =&x^4 - 2x^3 - \frac{1}{2}x^2 + x - \frac{3}{16} = \left(x-\frac{1}{2}\right)\left(x^3 - \frac{3}{2}x^2 - \frac{5}{4}x + \frac{3}{8}\right) =\\ =&\left(x-\frac{1}{2}\right)\left[x^2\left(x-\frac{3}{2}\right) - \frac{5}{4}\left(x-\frac{3}{10}\right)\right] \leqslant 0\end{aligned}$$

as we have $\frac{3}{10} < \frac{1}{2} \leqslant x \leqslant \frac{1}{2}\sqrt{2} < \frac{3}{2}$. Therefore, $S_{AS_1S_2} \leqslant \frac{3}{16}$.

3. *Let $p(n)$ denote the number of all n-digit positive integers containing only the digits 1, 2, 3, 4, 5 and such that every two adjacent digits differ by at least 2. Prove that for every positive integer n,*

$$5 \cdot 2.4^{n-1} \leqslant p(n) \leqslant 5 \cdot 2.5^{n-1}.$$

(Pavel Novotný)

Solution. Cutting off the last digit of a satisfactory $(n+1)$-digit integer yields a satisfactory n-digit integer. Notice how a satisfactory $(n+1)$-digit integer can be constructed from a satisfactory n-digit integer. If the last digit of the integer is 1, we can append any of the digits 3, 4, 5. If the last digit is 2, we can append 4 or 5; if it is 3, 1 or 5 can be appended; if it is 4, 1 or 2 can be appended; and, finally, in the case of 5, we can append any of the digits 1, 2, 3. Thus we can see that only the last digit matters. So now, let a_n denote the number of satisfactory n-digit integers ending in 1 or 5; similarly b_n for 2 or 4, and c_n for integers ending in 3. Then $p(n) = a_n + b_n + c_n$. Apparently, $a_1 = b_1 = 2$, $c_1 = 1$, $p(1) = 5 = 5 \cdot 2.4^0 = 5 \cdot 2.5^0$, $a_2 = 6$, $b_2 = 4$, $c_2 = 2$, $p(2) = 12 = 5 \cdot 2.4^1 < 5 \cdot 2.5^1$.

The above reasoning implies the recurrent formulae

$$a_{n+1} = a_n + b_n + 2c_n, \quad b_{n+1} = a_n + b_n, \quad c_{n+1} = a_n. \tag{1}$$

Hence it follows that $a_3 = 14$, $b_3 = 10$, $c_3 = 6$, $p(3) = 30 \in \langle 5 \cdot 2.4^2; 5 \cdot 2.5^2 \rangle$.

Using mathematical induction, we prove that for every $n \geqslant 3$, it holds that

$$a_n \geqslant 2.4^n, \quad b_n \geqslant \frac{2}{3} \cdot 2.4^n, \quad c_n \geqslant 2.4^{n-1}.$$

It indeed does for $n = 3$. If $a_n \geqslant 2.4^n$, $b_n \geqslant \frac{2}{3} \cdot 2.4^n$ and $c_n \geqslant 2.4^{n-1}$, then also

$$\begin{aligned} a_{n+1} = a_n + b_n + 2c_n \geqslant & 2.4^n + \frac{2}{3} \cdot 2.4^n + 2 \cdot 2.4^{n-1} = \\ & = 2.4^n \cdot \left(1 + \frac{2}{3} + \frac{5}{6}\right) = 2.5 \cdot 2.4^n > 2.4^{n+1}, \\ b_{n+1} = a_n + b_n \geqslant & 2.4^n + \frac{2}{3} \cdot 2.4^n = \frac{5}{3} \cdot 2.4^n > \frac{2}{3} \cdot 2.4^{n+1}, \\ c_{n+1} = a_n \geqslant & 2.4^n. \end{aligned}$$

It follows from the proved inequalities that

$$p(n) = a_n + b_n + c_n \geqslant 2.4^n + \frac{2}{3} \cdot 2.4^n + 2.4^{n-1} = (2.4 + 1{,}6 + 1) \cdot 2.4^{n-1} = 5 \cdot 2.4^{n-1}.$$

The latter inequality can be proved analogously; we will verify that for $n \geqslant 3$,

$$a_n \leqslant k \cdot 2.5^n, \quad b_n \leqslant k \cdot \frac{2}{3} \cdot 2.5^n, \quad c_n \leqslant k \cdot 2.5^{n-1}, \tag{2}$$

where k is a suitably chosen number. Then we will have

$$p(n) = a_n + b_n + c_n \leqslant k \cdot 2.5^{n-1} \cdot \left(2.5 + \frac{5}{3} + 1\right) = k \cdot 2.5^{n-1} \cdot \frac{31}{6} = 5k \cdot \frac{31}{30} \cdot 2.5^{n-1}.$$

Therefore, setting $k = \frac{30}{31}$, we get $p(n) \leqslant 5 \cdot 2.5^{n-1}$ for every $n \geqslant 3$.

It remains to prove, by mathematical induction, the inequalities (2) where $k = \frac{30}{31}$. They hold for $n = 3$. If (2) holds, we also have

$$\begin{aligned} a_{n+1} = a_n + b_n + 2c_n &\leqslant k \cdot 2.5^n \cdot \left(1 + \frac{2}{3} + \frac{4}{5}\right) = k \cdot 2.5^n \cdot \frac{37}{15} < k \cdot 2.5^{n+1}, \\ b_{n+1} = a_n + b_n &\leqslant k \cdot 2.5^n \cdot \left(1 + \frac{2}{3}\right) = k \cdot \frac{2}{3} \cdot 2.5^{n+1}, \\ c_{n+1} = a_n &\leqslant k \cdot 2.5^n. \end{aligned}$$

Other Solution. We will show that each of the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ defined in the above solution satisfies (as a consequence of the equalities (1)) the recurrent equation $x_{n+2} = 2x_{n+1} + 2x_n - 2x_{n-1}$, and so this equation is satisfied by the sequence $p(n) = a_n + b_n + c_n$ in question, which we will denote by (3).

Indeed, the first and third equalities of (1) give $a_{n+1} = a_n + b_n + 2a_{n-1}$, whence

$$b_n = a_{n+1} - a_n - 2a_{n-1}, \quad \text{so} \quad b_{n+1} = a_{n+2} - a_{n+1} - 2a_n.$$

Considering the second equality in (1), we thus get

$$a_{n+2} - a_{n+1} - 2a_n = b_{n+1} = a_n + b_n = a_n + (a_{n+1} - a_n - 2a_{n-1}).$$

Confronting the marginal expressions leads to the mentioned equality

$$a_{n+2} = 2a_{n+1} + 2a_n - 2a_{n-1}.$$

Triple substitution of $a_n = b_{n+1} - b_n$ into the equality $b_n = a_{n+1} - a_n - 2a_{n-1}$ yields

$$b_n = (b_{n+2} - b_{n+1}) - (b_{n+1} - b_n) - 2(b_n - b_{n-1}),$$

which can be rearranged to

$$b_{n+2} = 2b_{n+1} + 2b_n - 2b_{n-1}.$$

Finally, the sequence $\{c_n\}$ is merely a shifted sequence $\{a_n\}$, so

$$c_{n+2} = a_{n+1} = 2a_n + 2a_{n-1} - 2a_{n-2} = 2c_{n+1} + 2c_n - 2c_{n-1}.$$

Combining all of the three recurrent formulae, we get

$$p(n+2) = 2p(n+1) + 2p(n) - 2p(n-1). \tag{3}$$

Using mathematical induction, we will prove that for every $k \geqslant 1$,

$$2.4p(k) \leqslant p(k+1) \leqslant 2.5p(k). \tag{4}$$

The inequalities (4) hold for both $k = 1$ and $k = 2$. If (4) holds for all $k \in \{1, 2, \dots, n+1, n+2\}$, then

$$\begin{aligned} p(n+3) = 2\big(p(n+2) + p(n+1) - p(n)\big) \geqslant \\ \geqslant 2\Big(p(n+2) + p(n+1) - \frac{p(n+1)}{2.4}\Big) = 2\Big(p(n+2) + \frac{7p(n+1)}{12}\Big) \geqslant \\ \geqslant 2\Big(p(n+2) + \frac{7}{12} \cdot \frac{p(n+2)}{2.5}\Big) = \frac{74p(n+2)}{30} > 2.4p(n+2). \end{aligned}$$

Similarly,

$$\begin{aligned} p(n+3) \leqslant &2\Big(p(n+2) + p(n+1) - \frac{p(n+1)}{2.5}\Big) \leqslant \\ \leqslant &2\Big(p(n+2) + \frac{3}{5} \cdot \frac{p(n+2)}{2.4}\Big) = 2.5p(n+2). \end{aligned}$$

The equalities $5 \cdot 2.4^0 = p(1) = 5 \cdot 2.5^0$ and the inequalities of (4) imply that the inequality $5 \cdot 2.4^{n-1} \leqslant p(n) \leqslant 5 \cdot 2.5^{n-1}$ holds for every positive integer n.

Poznámka. The equation (3) is called *linear differential equation with constant coefficients*. The well-known recurrent formula $g_{n+1} = g_n \cdot q$ for geometric sequences indicates that the equation (3) could be satisfied by some geometric sequences, i. e., $p(n) = q^n$. Substituting into (3) yields the so-called *characteristic equation* for the common ratio q:

$$q^3 - 2q^2 - 2q + 2 = 0,$$

which has three real roots $q_1 \doteq -1.170\,086\,487$, $q_2 \doteq 0.688\,892\,182$, $q_3 \doteq 2.481\,194\,304$. It can be proved that every solution of the equation (3) is a linear combination of the sequences $\{q_1^n\}$, $\{q_2^n\}$ and $\{q_3^n\}$, i. e.,

$$p(n) = \alpha \cdot q_1^n + \beta \cdot q_2^n + \gamma \cdot q_3^n.$$

The coefficients α, β, γ can be determined from the system of equations

$$\alpha q_1 + \beta q_2 + \gamma q_3 = p(1) = 5, \ \ \alpha q_1^2 + \beta q_2^2 + \gamma q_3^2 = p(2) = 12, \ \ \alpha q_1^3 + \beta q_2^3 + \gamma q_3^3 = p(3) = 30.$$

Instead of the third equation, we could use $\alpha + \beta + \gamma = p(0) = 2$; the number $p(0)$ cannot be defined as the number of 0-digit integers, yet $p(0) = 2$ is taken to satisfy (3) with $n = 1$. Therefore, we get the approximation

$$p(n) \approx -0.063\,627\,546 q_1^n + 0.108\,637\,179 q_2^n + 1.954\,990\,367 q_3^n$$

for the terms of the examined sequence. (This approximation can be used for n up to around 20; for greater values of n, the rounding errors begin to take effect.)

4. *Find all functions $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ such that for all non-zero numbers x, y,*

$$x \cdot f(xy) + f(-y) = x \cdot f(x).$$

(Pavel Calábek)

Solution. Substituting $x = 1$ gives

$$f(y) + f(-y) = f(1).$$

Denoting $f(1) = a$, we have $f(-y) = a - f(y)$. Substituting $y = -1$, we get

$$x \cdot f(-x) + f(1) = x \cdot f(x),$$

i. e.,

$$x\big(a - f(x)\big) + a = x \cdot f(x),$$

hence

$$f(x) = \frac{a(x+1)}{2x} = \frac{a}{2}\Big(1 + \frac{1}{x}\Big).$$

Finally, we check that for any real number c, the function $f(x) = c(1 + 1/x)$ satisfies the conditions:

$$x \cdot f(xy) + f(-y) = x \cdot c\Big(1 + \frac{1}{xy}\Big) + c\Big(1 + \frac{1}{-y}\Big) = c\Big(x + \frac{1}{y} + 1 - \frac{1}{y}\Big) =$$
$$= c(x+1) = cx\Big(1 + \frac{1}{x}\Big) = x \cdot f(x).$$

Other Solution. Let us set $f(1) = a$. Substituting $y = -1$ into the given equation yields

$$xf(-x) + a = xf(x),$$

i. e.,

$$f(-x) = f(x) - \frac{a}{x}. \tag{1}$$

The given equation can be rearranged into the form

$$f(xy) + \frac{1}{x} \cdot f(-y) = f(x),$$

whence, using (1), we have

$$f(xy) + \frac{1}{x}\Big(f(y) - \frac{a}{y}\Big) = f(x),$$
$$f(xy) + \frac{f(y)}{x} - \frac{a}{xy} = f(x).$$

Interchanging x and y gives

$$f(yx) + \frac{f(x)}{y} - \frac{a}{yx} = f(y),$$

so, combining the last two equations, we get

$$\frac{f(x)}{y} - \frac{f(y)}{x} = f(y) - f(x),$$

and substituting $y = 1$ now gives

$$2f(x) = a\Big(1 + \frac{1}{x}\Big).$$

Again, we can easily verify that every function $f(x) = c(1 + 1/x)$ is a solution of the given functional equation.

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5. *Let I be the incenter of a triangle ABC. The circle passing through the vertex B and touches the line AI at I intersects the sides AB and BC at points P and Q, respectively. Let R be the intersection point of the line QI and the side AC. Prove that*

$$|AR| \cdot |BQ| = |PI|^2.$$

(Jaroslav Švrček)

Solution. Let α, β, γ denote the measures of the interior angles at the vertices A, B, C, respectively, of the triangle ABC, and let J be the intersection point of the line AI and the side BC. (Fig. 2). The inscribed angle PBI corresponds to the chord PI, while the inscribed angle QBI corresponds to the chord IQ, and since the measure of both of these angles is $\frac{1}{2}\beta$, the chords PI and IQ share the same length as well.

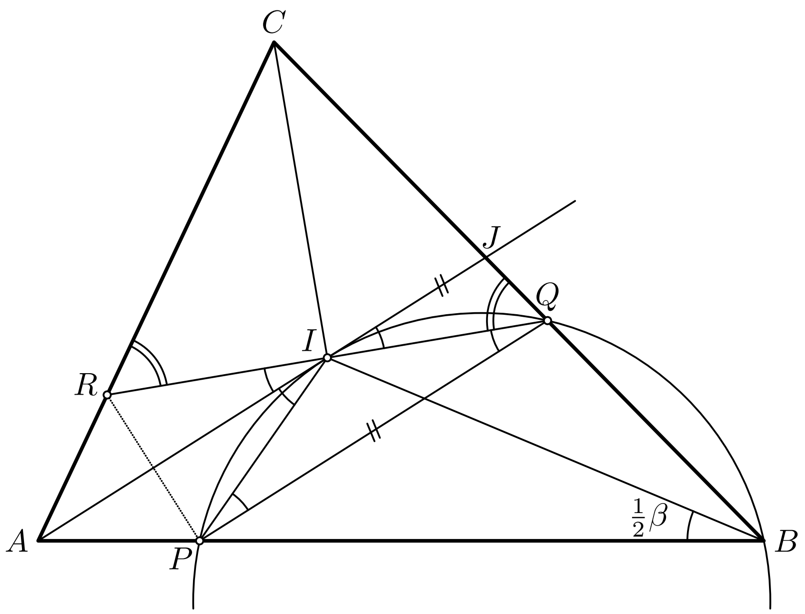

Fig. 2

Since the inscribed angle JIQ also corresponds to the chord IQ, its measure is also $\frac{1}{2}\beta$. It follows from the congruence of vertical angles that $|\angle RIA| = \frac{1}{2}\beta$. This measure is also shared by the inscribed angle PIA as it corresponds to the chord PI of equal length as IQ. Further, $|\angle RAI| = |\angle PAI| = \frac{1}{2}\alpha$. The triangles RIA and PIA are thus congruent by ASA, hence $|RI| = |PI|$.

Therefore, the measure of the angle QIB is

$$\begin{aligned}|\angle QIB| =& 180^\circ - |\angle AIB| - |\angle JIQ| = 180^\circ - \left(90^\circ + \frac{\gamma}{2}\right) - \frac{\beta}{2} = \\ =& 90^\circ - \left(\frac{\beta}{2} + \frac{\gamma}{2}\right) = \frac{\alpha}{2} = |\angle RAI|.\end{aligned}$$

The measure of the angle QIB could also be determined as follows: Since the inscribed angles AIP and IPQ correspond to chords of equal length, we have $|\angle AIP| = \frac{1}{2}\beta = |\angle IPQ|$. The congruence of alternate angles implies that $AI \parallel PQ$. Hence $|\angle QPB| = |\angle IAB| = \frac{1}{2}\alpha$, and so $|\angle QIB| = \frac{1}{2}\alpha$ as well since they are both inscribed angles corresponding to the chord QB.

Since $|\angle QIB| = |\angle RAI|$ and $|\angle QBI| = |\angle RIA|$, it follows that $AIR \sim IBQ$. Therefore, $|AR|/|RI| = |IQ|/|QB|$, so

$$|AR| \cdot |QB| = |RI| \cdot |IQ| = |PI|^2.$$

To prove similarity of the triangles AIR and IBQ, we could have made use of the fact that the triangle CRQ is isosceles, so its median coincides with its angle bisector.

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6. *In the real numbers, solve the following system of equations:*

$$\begin{aligned}\sin^2 x + \cos^2 y &= \tan^2 z,\\ \sin^2 y + \cos^2 z &= \tan^2 x,\\ \sin^2 z + \cos^2 x &= \tan^2 y.\end{aligned}$$

(Pavel Calábek)

Solution. Substituting $\cos^2 x = a$, $\cos^2 y = b$, $\cos^2 z = c$ leads to the system

$$\begin{aligned}1 - a + b &= \frac{1}{c} - 1,\\ 1 - b + c &= \frac{1}{a} - 1,\\ 1 - c + a &= \frac{1}{b} - 1,\end{aligned} \tag{1}$$

where $a, b, c \in (0, 1\rangle$.

Adding these equations together yields

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 6,$$

so the harmonic mean of the numbers a, b, c is $\frac{1}{2}$.

Multiplying the equations by the numbers c, a, and b, respectively, we get

$$\begin{aligned} c - ac + bc =& 1 - c, \\ a - ab + ac =& 1 - a, \\ b - bc + ab =& 1 - b, \end{aligned}$$

which sums to $2(a + b + c) = 3$. Therefore, the arithmetic mean of the numbers a, b, c is $\frac{1}{2}$, too. Since the arithmetic and harmonic means are equal, it must be that $a = b = c = \frac{1}{2}$. We can easily verify that this triple satisfies the system (1). The solutions of the original system are thus exactly the triples $(\frac{1}{4}\pi + \frac{1}{2}k\pi, \frac{1}{4}\pi + \frac{1}{2}l\pi, \frac{1}{4}\pi + \frac{1}{2}m\pi)$, where k, l, m are integers.

Jiné řešení. We use the substitution from the above solution. The system (1) is cyclic; if a triple (p, q, r) satisfies it, then so do the triples (q, r, p) and (r, p, q). Therefore, it suffices to find the solutions for which $a \geqslant b$, $a \geqslant c$, and all the other solutions can then be obtained by cyclic exchange.

Let $a \geqslant b$, $a \geqslant c$. It follows from the first equation that $1/c = 2 - a + b \leqslant 2$, so $c \geqslant \frac{1}{2}$. Similarly, it follows from the third equation that $1/b = 2 - c + a \geqslant 2$, so $b \leqslant \frac{1}{2}$, and thus $b \leqslant c$. ¿From the second equation, we have $1/a = 2 - b + c \geqslant 2$, so $a \leqslant \frac{1}{2}$. Altogether,

$$\frac{1}{2} \geqslant a \geqslant c \geqslant \frac{1}{2},$$

so $a = c = \frac{1}{2}$. Now, any of the equations of the system (1) yields $b = \frac{1}{2}$. Just as in the previous case, we verify that the found triple satisfies the system (1); so the solutions of the system are the triples $(\frac{1}{4}\pi + \frac{1}{2}k\pi, \frac{1}{4}\pi + \frac{1}{2}l\pi, \frac{1}{4}\pi + \frac{1}{2}m\pi)$, where k, l, m are integers.

First Round of the 62nd Czech and Slovak Mathematical Olympiad (December 6th, 2012)

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1. *There are two touching circles, $k_1(S_1, r_1$ and $k_2(S_2, r_2)$ in an rectangle $ABCD$ with $|AB| = 9$, $|BC| = 8$. Moreover k_1 touches AD and CD, while k_2 touches AB and BC.*

a) Prove $r_1 + r_2 = 5$.

b) What is the least and what is the greatest possible area of AS_1S_2?

(Pavel Novotný)

Solution. a) Let M and N be intersections of the line through S_1 parallel to AD. Analogously let K and L be intersections of the line through S_2 parallel to AB. Let P be the intersection of KL and MN (see Fig. 1). The Pythagoras theorem for S_1PS_2 gives

$$(r_1 + r_2)^2 = (8 - r_1 - r_2)^2 + (9 - r_1 - r_2)^2, \tag{1}$$

$$(r_1 + r_2)^2 - 34(r_1 + r_2) + 145 = 0,$$

$$(r_1 + r_2 - 5)(r_1 + r_2 - 29) = 0.$$

Since $2r_1 \leqslant 8$, $2r_2 \leqslant 8$, we have $r_1 + r_2 = 5$.

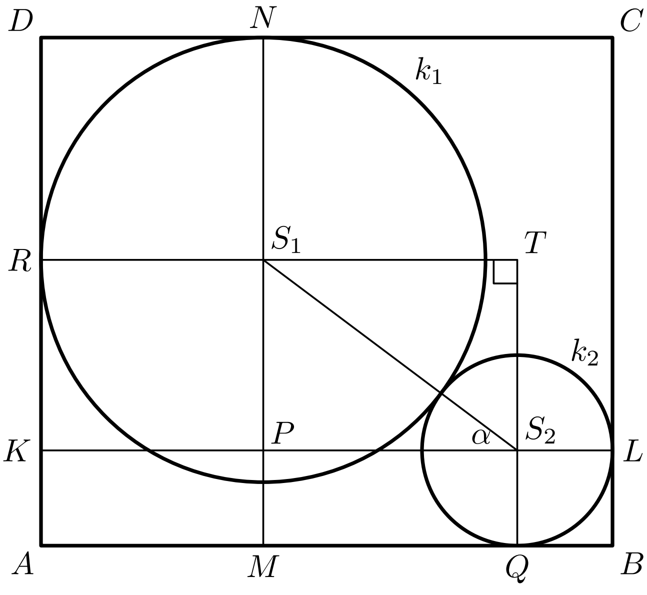

Fig. 1

b) Let Q be a foot of a perpendicular to AB from S_2, let R be a foot of a perpendicular to AD from S_1 and let T be the intersection of QS_2 and RS_1 (Fig. 1).

The area S of AS_2S_1 is given by the difference of the area of rectangle $AQTR$ and areas of right triangles AQS_2, AS_1R, and S_1S_2T:

$$\begin{aligned} S &= (9-r_2)(8-r_1) - \frac{1}{2}r_2(9-r_2) - \frac{1}{2}r_1(8-r_1) - \frac{1}{2}(9-r_1-r_2)(8-r_1-r_2) \\ &= 72 - 9r_1 - 8r_2 + r_1r_2 - \frac{9}{2}r_2 + \frac{1}{2}r_2^2 - 4r_1 + \frac{1}{2}r_1^2 - 36 + \frac{17}{2}(r_1+r_2) \\ &\quad - \frac{1}{2}(r_1+r_2)^2 \\ &= 36 - \frac{9}{2}r_1 - 4r_2 = 36 - \frac{9}{2}r_1 - 4(5-r_1) = 16 - \frac{1}{2}r_1, \end{aligned}$$

where we used $r_1 + r_2 = 5$. Further we know $2r_1 \leqslant 8$ and $2r_2 \leqslant 8$ which implies r_1, r_2 $1 \leqslant r_1, r_2 \leqslant 4$, thus

$$S = 16 - \frac{1}{2}r_1 \in \left\langle 14, \frac{31}{2} \right\rangle;$$

and the least possible value of the area is 14, for $r_1 = 4$ and $r_2 = 1$, and the greatest value possible is $\frac{31}{2}$, for $r_1 = 1$ and $r_2 = 4$.

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2. *The number 0 is written on each of the $n+1$ faces of an n-sided pyramid. In a step we choose a vertex and we increase by 1 each number on the faces, which contain the vertex. Show, that in such way, we cannot get number 1 written on each face.* (Peter Novotný)

Solution. Let b the sum of numbers on side faces of the pyramid, let a be the number on the base. After a step involving any base vertex, b increases or decreases by 2 and a increases or decreases by 1, that means the the value $V = b - 2a$ stays the same. If we choose for a step the apex, only b increases or decreases by n, thus V increases or decreases by n as well. Therefore V is in the process always divisible by n. But in the position with number 1 written on each side, the corresponding V is $n-2$, which in not divisible by n (as $n > 2$).

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3. *Find all real a, b, c, such that*

$$a^2 + b^2 + c^2 = 26, \quad a + b = 5 \quad \text{and} \quad b + c \geqslant 7.$$

(Pavel Novotný)

Solution. We show that the only solution is $a = 1$, $b = 4$ a $c = 3$.

Let $s = b + c \geqslant 7$. Substituting $a = 5 - b$ and $c = s - b$ the first condition gives

$$(5-b)^2 + b^2 + (s-b)^2 = 26,$$

thus

$$3b^2 - 2(s+5)b + s^2 - 1 = 0. \tag{5}$$

The equation has a real solution iff the discriminant $4(s+5)^2 - 12(s^2-1) \geqslant 0$. This yields $s^2 - 5s - 14 \leqslant 0$, or $(s+2)(s-7) \leqslant 0$. Since $s \geqslant 7$, there must be $s = 7$. If we substitute to (5) we get

$$3b^2 - 24b + 48 = 0;$$

with the only solution $b = 4$. Then $a = 1$ and $c = 3$.

Second Round of the 62nd Czech and Slovak Mathematical Olympiad (January 15th, 2013)

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1. *In a group of 21 different integers a sum of arbitrary eleven ones is greater than a sum of the remaining ten numbers.*

a) Prove that every considered number is greater than 100.

b) Find all such groups of 21 *different integers containing number* 101.

(Jaromír Šimša)

Solution. a) Let the numbers are $a_1 < a_2 < a_3 < \cdots < a_{21}$. Since they are integers, for every $i \in \{1, 2, \ldots, 20\}$ holds $a_{i+1} - a_i \geqslant 1$, and therefore $a_{i+10} - a_i \geqslant 10$ for every $i \in \{1, 2, \ldots, 11\}$.

The problem condition is fulfilled if and only if the sum

$$a_1 + a_2 + \cdots + a_{11} > a_{12} + a_{13} + \cdots + a_{21}. \tag{1}$$

This follows

$$a_1 > (a_{12} - a_2) + (a_{13} - a_3) + \cdots + (a_{21} - a_{11}) \geqslant 10 \cdot 10 = 100.$$

Since the least of the numbers is greater then 100, the other ones are greater than 100 too.

b) We have proved $a_1 \geqslant 101$. The other numbers are greater than 101. If the number 101 is in the group of positive integers, then $a_1 = 101$ holds. The strict inequality (1) gives

$$(a_{12} - a_2) + (a_{13} - a_3) + \cdots + (a_{21} - a_{11}) \leqslant a_1 - 1 = 100,$$

and because $a_{i+10} - a_i \geqslant 10$, the equality $a_{i+10} - a_i = 10$ holds for every $i \in \{2, 3, \ldots, 11\}$. It comes to pass if and only if the numbers $a_2, a_3, \ldots, a_{21}$ are consecutive integers.

The required group consists of number 101 and arbitrary 20 consecutive integers which are greater than 101. Then the difference of sums of 11 minimal numbers and 10 maximal numbers is 1.

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2. *Let A, B be sets of positive integers such that a sum of arbitrary two different numbers from A is in B and a ratio of arbitrary two different numbers from B (greater one to smaller one) is in A. Find the maximum number of elements in $A \cup B$.*

(Martin Panák)

Solution. Initially we will prove that the set A consists from at most two numbers. Suppose that three numbers $a < b < c$ belongs to the set A. Then the numbers $a+b < a+c < b+c$ are in B and therefore the number

$$\frac{b+c}{a+c} = 1 + \frac{b-a}{a+c};$$

have to be in A. This is contradiction because $0 < b-a < a+c$ and the number is not integer.

If the set B contains four numbers $k < l < m < n$, then the set A will contain three different numbers n/k, n/l, n/m. So the set B has at most three elements and $A \cup B$ has at most five elements.

We achieve the number 5 of elements if $A = \{a, b\}$, $B = \{k, l, m\}$, where $a < b$ and $l/k = m/l = a$, $m/k = b$. Then $b = a^2$ $(a \geqslant 2)$ and $a + a^2$ is one of the elements of B; the next two elements are either $a^2 + a^3$ and $a^3 + a^4$ or $1 + a$ and $a^2 + a^3$. Eg. sets $A = \{2, 4\}$, $B = \{3, 6, 12\}$ have five elements together.

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3. *Touching circles $k_1(S_1, r_1)$ and $k_2(S_2, r_2)$ lie in a right-angled triangle ABC with the hypotenuse AB and legs $AC = 4$ and $BC = 3$ in such way, that the sides AB, AC are tangent to k_1 and the sides AB, BC are tangent to k_2. Find radii r_1 a r_2, if $4r_1 = 9r_2$.* (Pavel Novotný)

Solution. The hypotenuse AB has length $AB = 5$ with respect to Pythagoras' theorem. Then for angles in the triangle there is $\cos\alpha = \frac{4}{5}$, $\cos\beta = \frac{3}{5}$,

$$\cot\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = 3,$$

$$\cot\frac{\beta}{2} = \sqrt{\frac{1+\cos\beta}{1-\cos\beta}} = 2.$$

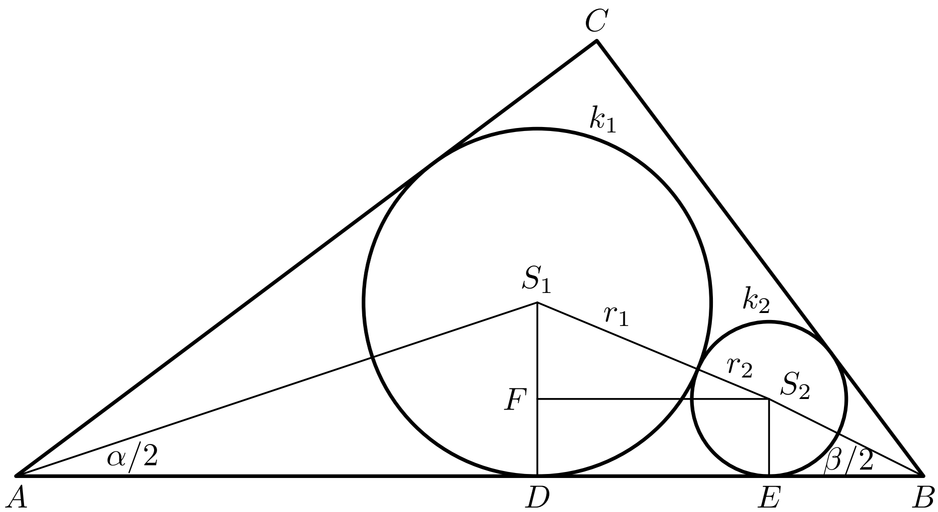

Fig. 1

Since both circles k_1, k_2 whole lie in the triangle ABC, they are externally tangent—in the opposite case the leg tangent to the smaller circle intersects the greater circle. Let circles k_1 and k_2 touch the side AB at points D resp. E and let point F be orthogonal projection of the point S_2 to the element S_1D (Fig. 1, under assumption is $r_1 > r_2$). Using Pythagoras' theorem for a triangle FS_2S_1 we obtain

$$(r_1 + r_2)^2 = (r_1 - r_2)^2 + DE^2,$$

which follows $DE = 2\sqrt{r_1 r_2}$.

An equality $AB = AD + DE + EB$ gives

$$c = r_1 \cot\frac{\alpha}{2} + 2\sqrt{r_1 r_2} + r_2 \cot\frac{\beta}{2} = 3r_1 + 2\sqrt{r_1 r_2} + 2r_2,$$

and since $r_1 = \frac{9}{4} r_2$ we obtain

$$\frac{27}{4} r_2 + 3r_2 + 2r_2 = 5,$$

which follows

$$r_2 = \frac{20}{47}, \quad r_1 = \frac{45}{47}.$$

Remark. Both circles lie really in the triangle ABC because an incircle of the triangle has diameter $\varrho = ab/(a + b + c) = 1$, while both values r_1, r_2 are less than 1.

---

4. *Prove that positive a, b, c are length of sides of a triangle if and only if a system of equations*

$$a(yz + x) = b(zx + y) = c(xy + z), \quad x + y + z = 1$$

with unknowns x, y, z has a solution in positive reals. (Tomáš Jurík)

Solution. Let a, b, c be positive numbers. We search a solution of the system of equations in the set of positive integers. Due to $x + y + z = 1$ the numbers x, y, z are in an interval $(0, 1)$. Substituting $z = 1 - x - y$ we obtain

$$a(y - xy - y^2 + x) = c(xy + 1 - x - y), \quad b(x - x^2 - xy + y) = c(xy + 1 - x - y),$$

these equations can be rewritten as

$$ay(1 - y) + ax(1 - y) = c(1 - x)(1 - y), \quad bx(1 - x) + by(1 - x) = c(1 - x)(1 - y).$$

Since $x < 1$, $y < 1$, we have

$$ay + ax = c - cx, \quad bx + by = c - cy.$$

From the previous two equations we obtain

$$x + y = \frac{2c}{a + b + c} \qquad \text{a} \qquad x - y = \frac{(b - a)(x + y)}{c} = \frac{2(b - a)}{a + b + c};$$

then we get formulas for x, y and finally by $z = 1 - x - y$ we find a formula for z too:

$$x = \frac{b+c-a}{a+b+c}, \quad y = \frac{c+a-b}{a+b+c}, \quad z = \frac{a+b-c}{a+b+c}. \tag{1}$$

The system has a solution in positive reals if and only if $b+c > a$, $c+a > b$, $a+b > c$ holds, what is equivalent with existence of a triangle with sides a, b, c.

We need not do check the values (1) due to equivalence of all rearrangements.

Other Solution. Here is an easier way to obtain (1). Since $x + y + z = 1$ we can rewrite the first part of the system as

$$a(1-y)(1-z) = b(1-z)(1-x) = c(1-x)(1-y). \tag{2}$$

Dividing $(1-x)(1-y)(1-z)$ (what is nonzero, even positive) we obtain equivalent system

$$\frac{a}{1-x} = \frac{b}{1-y} = \frac{c}{1-z}.$$

If s is common (positive) value of three previous fractions, we can easy get

$$x = 1 - \frac{a}{s}, \quad y = 1 - \frac{b}{s}, \quad z = 1 - \frac{c}{s}, \tag{3}$$

which substituting to the equation $x + y + z = 1$ gives

$$s = \frac{a+b+c}{2}.$$

It follows from (3) that this s yields the formulae (1), and then the proof of the problem statement.

Final Round of the 62nd Czech and Slovak Mathematical Olympiad (March 18–19, 2013)

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1. *Find all pairs of integers a, b such that*

$$\frac{a^2+1}{2b^2-3} = \frac{a-1}{2b-1}.$$

(Pavel Novotný)

Solution. Obviously $a \neq 1$, thus we can rewrite the equation as

$$\frac{a^2+1}{a-1} = \frac{2b^2-3}{2b-1}. \tag{1}$$

The numerator of the fraction on the left is positive, the numerator on the right is negative just for $b \in \{-1, 0, 1\}$.

For $b = -1$ we get $3a^2 - a + 4 = 0$, which has no real solution.

Similarly for $b = 0$ we get $a^2 - 3a + 4 = 0$ which has no real solution either.

For $b = 1$ we get $a^2 + a = a(a+1) = 0$, with solutions $a \in \{0, -1\}$. Thus pairs $(0, 1)$ and $(-1, 1)$ are solutions of the problem.

Let us further assume $2b^2 - 3 > 0$, and let us find out with which numbers we can reduce the fractions in (1).

If some integer n divides both a^2+1 and $a-1$, it divides $a^2+1-(a+1)(a-1) = 2$ as well. Similarly if n divides both $2b^2-3$ and $2b-1$, it divides $(2b-1)(2b+1) - 2(2b^2-3) = 5$.

Thus there are four possibilities to fulfill the equation in (1).

(i) $a^2 + 1 = 2b^2 - 3$ and $a - 1 = 2b - 1$, which has no real solution.

(ii) $a^2 + 1 = 2(2b^2 - 3)$ and $a - 1 = 2(2b - 1)$; substituting $a = 4b - 1$ into the first equation we get $3b^2 - 2b + 2 = 0$, with no real solutions.

(iii) $5(a^2 + 1) = 2b^2 - 3$ and $5(a - 1) = 2b - 1$, with solution $a = 0$, $b = -2$.

(iv) Finally $5(a^2 + 1) = 2(2b^2 - 3)$ and $5(a - 1) = 2(2b - 1)$ with solutions $a = -1$, $b = -2$ and $a = 7$, $b = 8$.

Thus there are five solutions of the problem:

$$(0, 1),\ (-1, 1),\ (0, -2),\ (-1, -2),\ (7, 8).$$

2. *Each of n Robin Hoods ($n \geqslant 3$) robbed some coins. Together they have earned $100n$ coins. They have decided to cut the loot in a following way: in one step one Robin can take his two coins and give to some other two Hoods, one coin each. Find all positive integers $n \geqslant 3$ for which they can split the loot in equal parts (100 coins each).* (Ján Mazák)

Solution. Let z_i denotes through the process the number of coins of ith Robin Hood.

Let $n = 3$. After any step, $z_1 - z_2$ modulo 3 does not change. That means for $z_1 = 101$, $z_2 = 100$, and $z_3 = 99$ (out of many choices), z_1 will never be the same as z_2. Thus $n = 3$ is not a solution.

Now we show that for each $n \geqslant 4$ any initial z_i the loot can be split in equal parts.

Let $s = \sum |z_i - 100|$. We decrease number s as long as it can be done in the way, that some of the outlaws with maximal count of coins gives his two coins to (some of the) Hoods, with the minimal count of coins. If s can be reduced to 0 in this way, we are done.

If $s \neq 0$, some of the outlaws has $100 - k$ coins ($k > 0$), k outlaws have 101 coins each and all the others have 100 coins each. If $k \geqslant 2$, we decrease s in two steps:

$$100 - k, 101, 101 \longrightarrow 100 - k + 1, 102, 99 \longrightarrow 100 - k + 2, 100, 100.$$

If k is even, then after $\frac{1}{2}k$ such "double" steps every outlaw will have 100 coins each. If k is odd then we end in the state in which one of the outlaws has 99 coins, one has 101 coins and all the others have 100 coins. We finish as follows:

$$99, 100, 100, 101 \longrightarrow 99, 101, 101, 99 \longrightarrow 99, 102, 99, 100 \longrightarrow 100, 100, 100, 100.$$

3. *Given a parallelogram $ABCD$ with center S, denote by O the incenter of triangle ABD and by T the point of contact of the incircle of triangle ABD with the diagonal BD. Prove that lines OS and CT are parallel.* (Jaromír Šimša)

Solution. Denote the lengths of AB, AD, and BD by a, b, and c, respectively. If $a = b$ then both OS and CT coincide with AC and the conclusion is trivial. Suppose $a > b$ (the case $b > a$ being completely analogous).

Let T' be the reflection of T in S (Fig. 1). As $CT \parallel AT'$, it suffices to prove $OS \parallel AT'$. Denoting by E the intersection of AO and the diagonal BD we may as well prove

$$\frac{AO}{OE} = \frac{T'S}{SE} \tag{1}$$

(note that since $a > b$, points T', S, E, and T lie on the diagonal BD in this order). We express both ratios in terms of a, b, c.

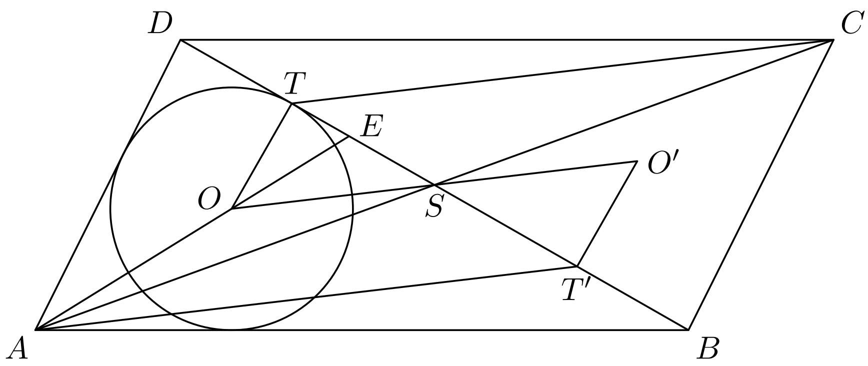

Fig. 1

First, it is well-known that

$$DT = \frac{b+c-a}{2}, \quad \text{and hence} \quad T'S = TS = \frac{c}{2} - \frac{b+c-a}{2} = \frac{a-b}{2}.$$

Next, the Angle Bisector Theorem in triangles ABD and AED implies

$$BE : ED = AB : AD \quad \text{and} \quad AO : OE = AD : DE$$

which in turn gives

$$BE = \frac{ac}{a+b} \quad \text{and} \quad DE = \frac{bc}{a+b},$$
$$SE = BE - BS = \frac{ac}{a+b} - \frac{c}{2} = \frac{c(a-b)}{2(a+b)},$$
$$\frac{AO}{OE} = \frac{AD}{DE} = \frac{b}{\frac{bc}{a+b}} = \frac{a+b}{c}.$$

Finally for the right-hand side we calculate

$$\frac{T'S}{SE} = \frac{\frac{a-b}{2}}{\frac{c(a-b)}{2(a+b)}} = \frac{a+b}{c}$$

which finishes the proof.

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4. *There is written a number N (in the decimal representation) on the board. In a step we erase the last digit c and instead of the number m, which is now left on the board, we write number $|m-3c|$ (for example, if $N = 1\,204$ was written on the board, then after the step there will be $120 - 3 \cdot 4 = 108$). We continue until there is a one-digit number on the board. Find all positive integers N such that after a finite number of steps number 0 is left on the board.* (Peter Novotný)

Solution. Let us find N, which lead to zero on the board after only one step. Obviously $|m-3c| = 0$ iff $m = 3c$, which is $N = 10m + c = 31c$. All such N are of the form $N = 31c$, $c \in \{1, 2, \cdots, 9\}$.

We show that the solution of the problem are exactly all multiples of 31. Since $c = N - 10m$, there is $m - 3c = 31m - 3N$, that is the divisibility by 31 is preserved in the step. Now we show, that a multiple of 31 actually decreases in the step. We have already shown that for $N \leqslant 31 \cdot 9$. Let $N = 31k$, where $k \geqslant 10$. Then $m \geqslant 31$, $m - 3c > 0$, thus $|m - 3c| = 31m - 3N < 4N - 3N = N$ and we are done.

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5. *Let $ABCD$ be a parallelogram such that the projections K, L of D onto the sides AB, BC, respectively, are their interior points. Prove that $KL \parallel AC$ if and only if*

$$\angle BCA + \angle ABD = \angle BDA + \angle ACD.$$

(Ján Mazák)

Solution. Alternate angles ABD and CDB are equal (Fig. 2), hence $\angle BCA + \angle ABD + \angle BDA + \angle ACD = 180°$. The equality $\angle BCA + \angle ABD = \angle BDA + \angle ACD$ thus holds if and only if

$$\angle BCA + \angle ABD = 90°. \tag{1}$$

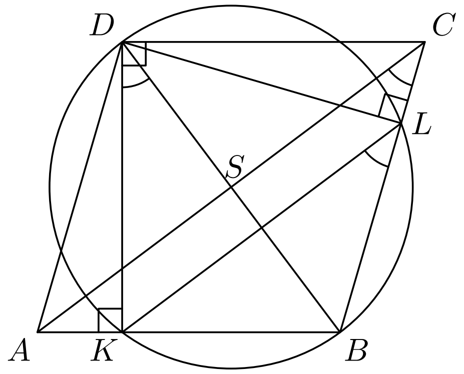

Fig. 2

Points K and L lie on a circle with diameter BD. Hence the inscribed angles BDK and BLK are equal and (due to equal alternate angles ABD and CDB)

$$\angle BLK + \angle ABD = \angle BDK + \angle CDB = 90°.$$

Lines KL and AC are parallel if and only if $\angle BLK = \angle BCA$ which is by the last equality equivalent to (1). The equivalence is thus proven.

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6. *Find all real p such that the inequality*

$$\sqrt{a^2 + pb^2} + \sqrt{b^2 + pa^2} \geqslant a + b + (p-1)\sqrt{ab}$$

holds for any real a and b. (Jaromír Šimša)

Solution. If $a = b = 1$ the parameter $p > 0$ has to satisfy:

$$2\sqrt{p+1} \geqslant p + 1,$$
$$2 \geqslant \sqrt{p+1},$$
$$p \leqslant 3.$$

We show that for $p \in (0,3\rangle$ the inequality holds for any real a and b.

If $p \in (0,1\rangle$ the inequality holds trivially:

$$\sqrt{a^2+pb^2} > a, \quad \sqrt{b^2+pa^2} > b \quad \text{a} \quad (p-1)\sqrt{ab} \leqslant 0.$$

Let further be $p \in (1,3\rangle$. The left hand side, LHS, of the inequality can be understood as the sum of the lengths of vectors $(a, b\sqrt{p})$ and $(b, a\sqrt{p}) \in \mathbb{R}^2$, According to the triangle inequality then

$$\begin{aligned} LHS = \sqrt{a^2+pb^2} + \sqrt{b^2+pa^2} &= |(a, b\sqrt{p})| + |(b, a\sqrt{p})| \\ &\geqslant |(a+b, (a+b)\sqrt{p})| = (a+b)\sqrt{1+p}. \end{aligned} \tag{1}$$

For the RHS we have (with the help of AM-GM inequality)

$$RHS = a+b+(p-1)\sqrt{ab} \leqslant a+b+(p-1)\frac{a+b}{2} = \frac{(p+1)(a+b)}{2}.$$

Now $LHS \geqslant RHS$ evidently, because even stronger inequality

$$(a+b)\sqrt{p+1} \geqslant \frac{(p+1)(a+b)}{2}$$

is equivalent to $\sqrt{p+1} \leqslant 2$, which is obviously satisfied for any $p \in (1,3\rangle$.

Remark. We can get (1) using Cauchy-Schwarz inequality for pairs $(a, b\sqrt{p})$ and $(1, \sqrt{p})$:

$$a + pb \leqslant \sqrt{a^2+pb^2} \cdot \sqrt{1+p},$$

which implies

$$\sqrt{a^2+pb^2} \geqslant \frac{a+pb}{\sqrt{1+p}}, \qquad \sqrt{b^2+pa^2} \geqslant \frac{b+pa}{\sqrt{1+p}},$$

summing last two inequalities we get (1) and analogously for the second inequality.

NFKJ
NADAČNÍ FOND KARLA JANEČKA

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