2016

65th Czech and Slovak Mathematical Olympiad

Translated into English by
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First Round of the 65th Czech and Slovak Mathematical Olympiad
Problems for the take-home part
(October 2015)

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1. *Some objects are in each of four rooms. Let $n \geqslant 2$ be an integer. We move one n-th of objects from the first room to the second one. Then we move one n-th of (the new number of) objects from the second room to the third one. Then we move similarly objects from the third room to the fourth one and from the fourth room to the first one. (We move the whole units of objects only.) Finally the same number of the objects is in every room. Find the minimum possible number of the objects in the second room. For which n does the minimum come?*

(Vojtech Bálint, Michal Rolínek)

Solution. Let us compute backwards. Firstly we find the number of the objects in two rooms before the move. Let a and b be number of the objects in the rooms A and B before the move. This number after the move we denote by a' and b'. By the conditions

$$a' = \frac{n-1}{n}a, \qquad b' = b + \frac{1}{n}a$$

holds. From the first equation and an identity $a + b = a' + b'$ we obtain

$$a = \frac{n}{n-1}a', \qquad b = b' - \frac{1}{n-1}a'.$$

Now let M be the final number of the objects in every room after the fourth move. By this identity we can compute the initial number of objects in every room in terms of M and n:

Finally:	M	M	M	M
before $4 \to 1$:	$\dfrac{n-2}{n-1}M,$	$M,$	$M,$	$\dfrac{n}{n-1}M;$
before $3 \to 4$:	$\dfrac{n-2}{n-1}M,$	$M,$	$\dfrac{n}{n-1}M,$	$M;$
before $2 \to 3$:	$\dfrac{n-2}{n-1}M,$	$\dfrac{n}{n-1}M,$	$M,$	$M;$
before $1 \to 2$:	$\dfrac{n(n-2)}{(n-1)^2}M,$	$\dfrac{(n-1)^2+1}{(n-1)^2}M,$	$M,$	$M.$

Since the number of objects in the first room was positive, $n \geqslant 3$ holds. Now we can easily find the minimum of

$$V_2 = \frac{(n-1)^2+1}{(n-1)^2} M.$$

The difference between numerator and denominator is 1, so the fraction is irreducible. Since V_2 is integer it must be $M = k(n-1)^2$ for proper k, therefore $V_2 = k\big((n-1)^2+1\big)$. For $n \geqslant 3$ we can estimate $(n-1)^2+1 \geqslant 5$, so $V_2 \geqslant 5$ too. Using $n = 3$, $k = 1$ and $M = 4$ we obtain $V_2 = 5$ and we can easily check that the quadruple $(3, 5, 4, 4)$ satisfies the problem: it transforms to quadruple $(2, 6, 4, 4)$, then $(2, 4, 6, 4)$, after that $(2, 4, 4, 6)$ and finally $(4, 4, 4, 4)$. So the minimal numbers of objects in the second room is 5 and we can obtain it only for $n = 3$ because for $n \geqslant 4$ is $V_2 \geqslant 3^2 + 1 = 10$.

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2. *Find the least real m such that there exist reals a and b for which the inequality*

$$|x^2 + ax + b| \leqslant m$$

holds for all $x \in \langle 0, 2\rangle$. (Leo Boček)

Solution. Notice that no negative number m satisfies the problem evidently (absolute value is non-negative number).

Now we interpret the problem geometrically. A graph of some function $y = x^2 + ax + b$ lies in a horizontal strip between lines $y = +m$ and $y = -m$ and in the interval $\langle 0, 2\rangle$. Our aim is to find the closest strip which contains the graph of such quadratic function in the interval $\langle 0, 2\rangle$.

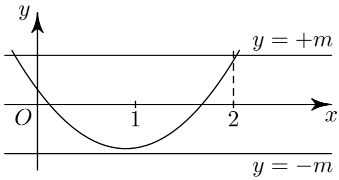

Fig. 1

The function

$$f(x) = (x-1)^2 - \tfrac{1}{2} = x^2 - 2x + \tfrac{1}{2},$$

seems to be a good candidate for such the closest strip. Such function has $a = -2$, $b = \frac{1}{2}$ and it satisfies (to be shown bellow) inequalities $-\frac{1}{2} \leqslant f(x) \leqslant \frac{1}{2}$.

Really, this inequalities are equivalent to the inequalities $0 \leqslant (x-1)^2 \leqslant 1$, which are evidently fulfilled for $x \in \langle 0, 2\rangle$. Quadratic function $f(x) = x^2 - 2x + \frac{1}{2}$ thus satisfies the conditions of the problem for $m = \frac{1}{2}$.

In the second part of the solution we will show that there is no quadratic function satisfying the problem for any $m < \frac{1}{2}$.

The crucial fact will be that at least one from differences $f(0) - f(1)$ and $f(2) - f(1)$ is greater or equal to 1 for an arbitrary function $f(x) = x^2 + ax + b$. This fact will imply that width of the closest strip will be greater or equal to 1 an this will exclude the values $m < \frac{1}{2}$. For $f(0) - f(1) \geqslant 1$ we obtain the desired estimate $2m \geqslant 1$ easily from the well-known triangle inequality $|a - b| \leqslant |a| + |b|$:

$$1 \leqslant |f(0) - f(1)| \leqslant |f(0)| + |f(1)| \leqslant 2m.$$

Similarly we estimate for $f(0) - f(1) \geqslant 1$.

Now it remains to verify at least one from inequalities $f(0) - f(1) \geqslant 1$ and $f(2) - f(1) \geqslant 1$ for arbitrary $f(x) = x^2 + ax + b$. The values

$$f(0) = b, \quad f(1) = 1 + a + b, \quad f(2) = 4 + 2a + b,$$

yields

$$\begin{aligned} f(0) - f(1) &= -1 - a \geqslant 1 \quad &\Leftrightarrow \quad a \leqslant -2, \\ f(2) - f(1) &= 3 + a \geqslant 1 \quad &\Leftrightarrow \quad a \geqslant -2. \end{aligned}$$

So at least one from inequalities $f(0)-f(1) \geqslant 1$ and $f(2)-f(1) \geqslant 1$ is true (regardless of the choice a, b).

Conclusion. The desired minimal value of m is $\frac{1}{2}$.

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3. *Let ABC be a right-angled triangle with a hypotenuse AB and longer leg BC. Let D be a foot of an altitude from the vertex C. Circle k with the center D and the radius CD intersects the leg BC in a point Q and line AB in points E and F ($E \neq F$), where F is a point on the hypotenuse AB. Segment QE intersects the leg AC in a point P. Prove that $PE = QF$.* (Jaroslav Švrček)

Solution. The circle k is the Thales' circle with the diameter EF and the center D. A triangle EFC is the isosceles right-angled triangle, so $EC = EF$. We will show that triangles EPC and FQC are congruent, which will prove the statement of the problem.

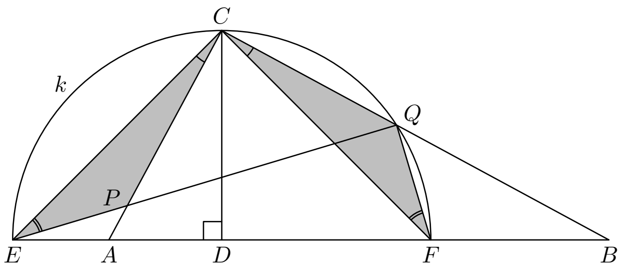

Fig. 2

Angles CEQ and CFQ are congruent as they are inscribed angles subtended the chord CQ of the circle k. Both angles ECF and ACB are congruent (right angles), so their remaining non-overlapping parts (angles $ECF = ECP$ and $ACB = FCQ$) are also congruent. This proves, that triangles EPC and FQC are congruent by A–S–A.

4. *Nela and Jane choose positive integer k and then play a game with a 9×9 table. Nela selects in every of her moves one empty unit square and she writes 0 to it. Jane writes 1 to some empty (unit) square in every her move. Furthermore k Jane's moves follows each Nela's move and Nela starts. If sum of numbers in each row and each column is odd anytime during the game, Jane wins. If girls fill out the whole table (without Jane's win), Nela wins. Find the least k such that Jane has the winning strategy.* (Michal Rolínek)

Solution. Let us show at first that Jane wins for $k = 3$. Let us assume 3×3 squares A_1, A_2 and A_3 (see the picture). We will call the 3×3 square *covered* if just one 1 is in each its row and column. If Jane covers squares A_1, A_2 and A_3 without writing to other squares, she wins, because sums in all rows and columns are odd number 1.

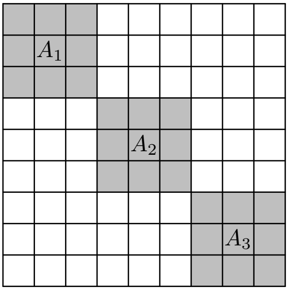

Fig. 3

It is obvious that if at most one 0 (and no 1) is written in any 3×3 square after Nela's move, Jane can cover this square because of $k = 3$. Jane has the following strategy: If Nela writes 0 to any uncovered square A_1, A_2 or A_3, Jane covers it immediately. In the opposite case Jane covers any of the uncovered 3×3 squares. Jane thus wins after three her triples of moves.

We will show that Nela has winning strategy for $k \in \{1, 2\}$. Let us realize that if Jane has some winning move, just 8 rows and 8 columns have odd sum before Jane's move, and the winning Jane's move is writing 1 to the intersection of the only one "even" row with the only one "even" column. This implies that if Jane has winning move, this move is unique.

Now it is obvious Nela's winning strategy for $k = 1$. If Jane has winning move after her move, Nela writes 0 to this square and Jane looses her unique chance for win. In the opposite case Nela writes 0 to some empty square. This move doesn't change parity of sums in rows and columns and Jane still hasn't winning move. This strategy allows Nela to fill out whole table without giving Jane chance to win.

In the case $k = 2$ Nela will use the same strategy as for $k = 1$. This strategy doesn't give Jane chance to win in her first move. In the second move Jane can't win because after that move the table contains even number of 1's which excludes possibility to be odd number 1's in every of (odd number) nine rows.

Conclusion. The least value k, for which Jane has winning strategy, is $k = 3$.

5. *Let ABC be a triangle with the shortest side BC. Let X, Y, K, L be a points on sides AB, AC and on ray's opposite to rays BC, CB respectively such that $BX = BK = BC = CY = CL$. Line KX intersects line LY in a point M. Prove that centroid of a triangle KLM coincides with incentre of the triangle ABC.* (Tomáš Jurík)

Solution. Since ABC is an external angle of the isosceles triangle XKB with the apex B (see the picture), a line KX is parallel to a bisectrix of the angle ABC.

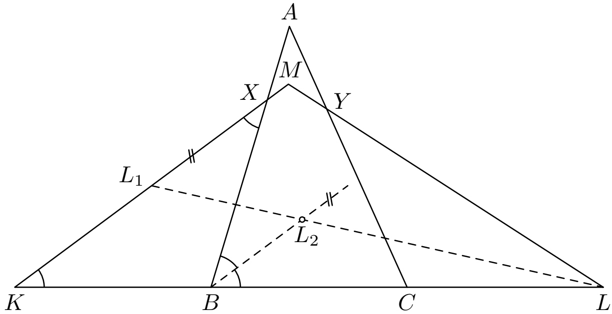

Fig. 4

The ratio $LB : LK = 2 : 3$ yields, that the bisectrix of ABC meets the centroid of the triangle KLM. If we denote LL_1 its median and L_2 its intersection with the bisectrix of ABC then we obtain

$$\frac{LL_2}{LL_1} = \frac{LB}{LK} = \frac{2}{3}.$$

from similarity of the triangles $\triangle LBL_2 \sim \triangle LKL_1$ (by A–A). So the point L_2 divides the median LL_1 in the same ratio as the centroid an therefore it is the centroid of the triangle KLM.

If follows from the symmetry of the problem, that bisectrix of the angle BCA meets the centroid of the triangle KLM. And the fact, that intersection of the bisectrices is the incentre, proves the claim of the problem.

6. *A product*

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

is written on a blackboard. For which positive integers $n \geqslant 2$ can we append the exclamation mark to some factors and change it to factorials in such a way that the final product will be a square? (Michal Rolínek)

Solution. Let us denote $v_p(n)$ the highest power of a prime p which divides positive integer n. This function has obviously the following properties:

– For all primes p and positive integers n is $v_p(n)$ non-negative integer.

– For all positive integers m, n and all primes p is $v_p(mn) = v_p(m) + v_p(n)$.
– For all primes p is $v_p(p!) = v_p(p) = 1$.
– For all primes p is $v_p((p+1)!) = 1$, $v_p(p+1) = 0$.
– For all primes p and all positive integers $n < p$ is $v_p(n!) = v_p(n) = 0$.
– Positive integer n is a square if and only if $v_p(n)$ is even (including zero) for all primes p.

Let us denote $S = n!$ the initial value of the product on the table and S' its final value after adding factorials. We can easy to see from the properties of v_p that for n is equal to any prime p we obtain $v_p(S) = v_p(p!) = 1$ and $v_p(S') = 1$, because adding factorials does not change the amount of the prime p $(= n)$ in the final product on the blackboard. The number $v_p(S')$ is then odd and therefore S' is not a square.

Let us assume that n is a composite number (so $n \geqslant 4$) in whole of the following part. We will show that we can add factorials in such a way that the final product

$$S' = f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n,$$

will be a square, where f_k is either k or $k!$ for all k. It is equivalent to $v_p(S')$ is even for all primes p. Since n is not a prime, only primes less than n occur in the product S'. As every such primes p are not in factors $f_1, f_2, \ldots, f_{p-1}$ and the prime p occurs in f_p only once, the final power $v_p(S')$ is the same as in a "reduced" product

$$p \cdot f_{p+1} \cdot f_{p+2} \cdot \ldots \cdot f_n. \tag{1}$$

How can we provide that every prime $p < n$ will occur in the corresponding product (1) with even power? Since in the second factor f_{p+1} from (1) occurs the prime p either once (in the case $f_{p+1} = (p+1)!$) or the prime p does not occure (if $f_{p+1} = p+1$), we can provide "good" occurrence of p by choice of f_{p+1} independently on succeeding values $f_{p+2}, \ldots, f_n$.*

Foregoing analysis gives us construction of the required choice of factorials. Initially we choose $f_k \in \{k, k!\}$ arbitrarily for all $k \leqslant n$ such that $k-1$ is not a prime. The other f_k, it is f_{p+1}, where p is arbitrary prime less than n, will be chosen "backwards", from the biggest such prime p to the smallest prime $p = 2$.** For the biggest unchosen f_{p+1} we find parity of $v_p(f_{p+2} \ldots f_n)$, in odd case we choose $f_{p+1} = p+1$, in even case we choose $f_{p+1} = (p+1)!$ and so on.

This finishes the construction of S' and solution of the problem too.

Conclusion. Desired $n \geqslant 2$ are all composite numbers.

* It is correct also for a prime $p = n-1$, where a factor $f_{p+1} = f_n$ is the last factor in (1); in this case we choose $f_n = n!$.

** The choice of f_3 will be the last, it corresponds to the smallest prime $p = 2$.

First Round of the 65th Czech and Slovak Mathematical Olympiad (December 8th, 2015)

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1. Nice prime *is a prime equal to the difference of two cubes of positive integers. Find last digits of all nice primes.* (Patrik Bak, Michal Rolínek)

Solution. Firstly, let us note that $5^3 - 4^3 = 61$, $2^3 - 1^3 = 7$ and $3^3 - 2^3 = 19$ are nice primes, so 1, 7 and 9 belong to desired digits. We show that they are all desired digits.

Let $p = m^3 - n^3$ be a nice prime, where $m > n$ are positive integers. Second factor in rewriting

$$p = m^3 - n^3 = (m - n)(m^2 + mn + n^2),$$

is greater than 1, thus the first one is 1 and therefore $m = n + 1$. After substitution we obtain

$$p = 3n^2 + 3n + 1. \tag{1}$$

An estimate $3n^2 + 3n + 1 > 6$ gives that the prime p is odd and greater that 5. This excludes 0, 2, 4, 5, 6 a 8 as the last digits and 3 stays the only remaining digit to exclude.

It is sufficient to find remainders of the numbers $3n^2 + 3n + 1$ after division by 5. For remainders 0, 1, 2, 3 and 4 of n we obtain remainders 1, 2, 4, 2, 1 of (1) which ones really exclude the last digit 3.

Answer. The last digits of the nice primes are 1, 7 and 9.

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2. *Positive real numbers a, b, c, d satisfy equalities*

$$a = c + \frac{1}{d} \quad \textit{and} \quad b = d + \frac{1}{c}.$$

Prove an inequality $ab \geqslant 4$ and find a minimum of $ab + cd$. (Jaromír Šimša)

Solution. To prove the inequality $ab \geqslant 4$ we substitute from the equalities. We so obtain an estimate

$$ab = \left(c + \frac{1}{d}\right)\left(d + \frac{1}{c}\right) = cd + 1 + 1 + \frac{1}{cd} \geqslant 4,$$

where we use in the last inequality well-known fact that $x + 1/x \geqslant 2$ holds for all positive reals $x = cd > 0$.

To find the minimum we use similar way. Substitution for a and b yields

$$ab + cd = \left(2 + cd + \frac{1}{cd}\right) + cd = 2 + 2cd + \frac{1}{cd}.$$

Now we use an inequality $x + y \geqslant 2\sqrt{xy}$ which holds true for any non-negative reals x, y. The choice $x = 2cd$, $y = 1/cd$ follows

$$2cd + \frac{1}{cd} \geqslant 2\sqrt{2}.$$

Now we see that $ab + cd \geqslant 2(1 + \sqrt{2})$. To prove that it is the desired minimum we find some a, b, c, d such that they makes an equality in the inequality.

The equality comes in the use inequality if and only if $x = y$, it is $2cd = 1/cd$. It is true e.g. for $c = 1$, $d = \sqrt{2}/2$ and for that values we find $a = 1 + \sqrt{2}$, $b = 1 + \sqrt{2}/2$. Such quadruple satisfies the desired equalities and it holds $ab + cd = 2(1 + \sqrt{2})$ too.

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3. *For a trapezoid $ABCD$ ($AB \parallel CD$) it holds $BC = AB + CD$. Prove that*

(i) *there is a point of a circle with diameter BC on the leg AD,*

(ii) *there is a point of a circle with diameter AD on the leg BC.*

(Josef Tkadlec)

Solution. (i) Let M, N be the centers of the legs BC, AD. We show that the point N lies on the circle with diameter BC.

A well-known identity yields

$$MN = \frac{AB + CD}{2} = \frac{1}{2} BC.$$

It means that the point N has the same distance from the center M of the circle with diameter BC as radius of that circle. So point M lies on that circle.

(ii) With respect to the given condition we can find a point E on the legs BC such that $|BE| = |AB|$ and $|EC| = |CD|$ (Fig. 1). We show that $\angle AED = 90°$ and so the point E is the desired point on the Thales' circle with diameter AD.

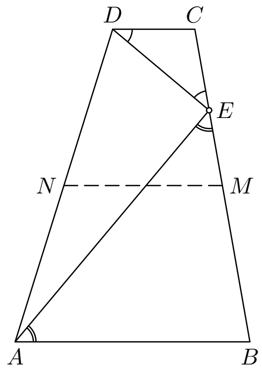

Fig. 1

The triangles ABE, ECD are isosceles and the lines AB and CD are parallel, thus the fact follows:

$$\begin{aligned}\angle AED &= 180^\circ - \angle AEB - \angle CED = \\ &= \tfrac{1}{2}\big((180^\circ - 2\angle AEB) + (180^\circ - 2\angle CED)\big) = \\ &= \tfrac{1}{2}(\angle ABE + \angle DCE) = 90^\circ.\end{aligned}$$

So we finished the second part.

Remark. If we start from proof that the triangle AED is right-angled one, we will become conscious of fact that its mutually perpendicular axes of sides AE and ED meet the center N of its circumscribed circle. It means that a triangle BCN is right-angled one too, so the circle with diameter BC meets the center N of the side AD.

Second Round of the 65th Czech and Slovak Mathematical Olympiad (January 12th, 2016)

1. *There are different positive integers written on the board. Their (arithmetic) mean is a decimal number, with the decimal part exactly* 0,2016. *What is the least possible value of the mean?* (Patrik Bak)

Solution. Let s be the sum, n the number and p the integer part of the mean of the numbers on the board. Then we can write

$$\frac{s}{n} = p + \frac{2\,016}{10\,000} = p + \frac{126}{625},$$

which gives

$$625(s - pn) = 126n.$$

Numbers 126 and 625 are coprime, thus $625 \mid n$. Therefore $n \geqslant 625$.

The numbers on the board are different, that is

$$p = \frac{s}{n} - \frac{126}{625} \geqslant \frac{1 + 2 + \cdots + n}{n} - \frac{126}{625} = \frac{n(n+1)/2}{n} - \frac{126}{625} \geqslant \frac{625+1}{2} - \frac{126}{625} > 312.$$

The integer p is thus at least 313 and the value of the mean at least 313,2016.

This value can be attained by numbers 1, 2, ... , 624 and 751. We get

$$\frac{1 + 2 + \cdots + 624 + 751}{625} = \frac{312 \cdot 625 + 751}{625} = 313 + \frac{126}{625} = 313{,}2016.$$

2. *On the unit square $ABCD$ is given point E on CD in such a way, that $|\angle BAE| = 60^\circ$. Further let X be an arbitrary inner point of the segment AE. Finally let Y be the intersection of a line, perpendicular to BX and containing X, with the line BC. What is the least possible length of BY?* (Michal Rolínek)

Solution. Let us consider the Thales circle over BY, which is circumscribed to BYX. This circle contains X and touches AB in B. Of all such circles is the one which touches AE (and it has to be in X) obviously the one with the least diameter (let us call the circle k). This circle is thus inscribed to the equilateral triangle $AA'F$ where A' is the image of A in the point symmetry with respect to B and F lies on the half line BC (see Fig. 1; there you can see one of the circles with smaller diameter than k as well). The center of k is the center of mass of the triangle $AA'F$, equilateral triangle with sides of length 2, thus the diameter of k is $BY = \frac{2}{3}\sqrt{3}$ and the corresponding X is a center of AF, that is it belongs to the segment AE, since $|AX| = 1 < |AE|$.

Answer. The least possible length of BY is $\frac{2}{3}\sqrt{3}$.

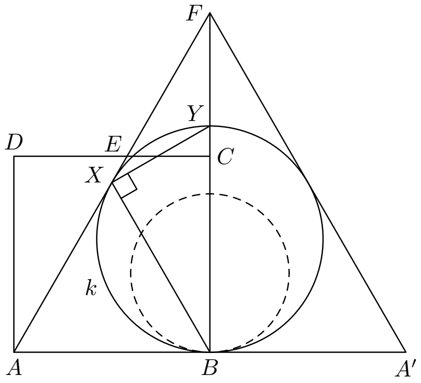

Fig. 1

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3. *In how many ways can you partition the set $\{1, 2, \ldots, 12\}$ into six mutually disjoint two-element sets in such a way that the two elements in any set are coprime?*
(Martin Panák)

Solution. No two even numbers can be in the same set (pair). Let us call partitions of $\{1, 2, \ldots, 12\}$ with this property, that is one even and one odd number in each pair, even-odd partitions. The only further limitations are, that 6 nor 12 cannot be paired with 3 or 9, and 10 cannot be paired with 5.

That means, that odd numbers 1, 7 and 11 can be paired with numbers 2, 4 ,6, 8, 10, 12, numbers 3 and 9 with 2, 4, 8, 10 and number 5 can be paired with 2, 4, 6, 8, 12. We cannot use the product rule directly, we distinguish two cases: 5 is paired with 6 or 12, in the second one 5 is paired with one of 2, 4, and 8. The possible pairings are $2 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = 144$ in the first case, $3 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 1 = 108$ in the second case. Together $144 + 108 = 252$ pairings.

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4. *Find the least real m, for which there exists real a, b such, that*

$$|x^2 + ax + b| \leqslant m(x^2 + 1)$$

holds for any $x \in \langle -1, 1 \rangle$. (Jaromír Šimša)

Solution. Let us assume that a, b, m satisfy the condition:

$$\forall x \in \langle -1, 1 \rangle \colon |f(x)| \leqslant m(x^2 + 1), \quad \text{kde} \quad f(x) = x^2 + ax + b.$$

Firstly we prove that at least one of the $f(1) - f(0) \geqslant 0$ and $f(-1) - f(0) \geqslant 0$ holds: for arbitrary $f(x) = x^2 + ax + b$ there is

$$f(0) = b, \quad f(1) = 1 + a + b, \quad f(-1) = 1 - a + b,$$

and

$$\max\bigl(f(1) - f(0), f(-1) - f(0)\bigr) = \max(1 + a, 1 - a) = 1 + |a| \geqslant 1.$$

Our assumption means $|f(1)| \leqslant 2m$, $|f(-1)| \leqslant 2m$ a $|f(0)| \leqslant m$. Consequently

$$1 \leqslant 1 + |a| = f(1) - f(0) \leqslant |f(1)| + |f(0)| \leqslant 2m + m = 3m, \tag{1}$$

or

$$1 \leqslant 1 + |a| = f(-1) - f(0) \leqslant |f(-1)| + |f(0)| \leqslant 2m + m = 3m. \tag{2}$$

In both cases we get $m \geqslant \frac{1}{3}$.

We show, that $m = \frac{1}{3}$ fulfills the problem conditions. For this m either (1) or (2) is equality, that is $a = 0$, $-f(0) = |f(0)|$ and $|f(0)| = m = \frac{1}{3}$, thus $b = f(0) = -\frac{1}{3}$.

We will verify, that the function $f(x) = x^2 - \frac{1}{3}$ pro $m = \frac{1}{3}$ satisfies the conditions of the problem: The inequality $|x^2 - \frac{1}{3}| \leqslant \frac{1}{3}(x^2+1)$ is equivalent with the inequalities

$$-\frac{1}{3}(x^2+1) \leqslant x^2 - \frac{1}{3} \leqslant \frac{1}{3}(x^2+1) \quad \text{or} \quad -x^2 - 1 \leqslant 3x^2 - 1 \leqslant x^2 + 1$$

which are equivalent to $0 \leqslant x^2 \leqslant 1$, which is evidently fulfilled on $\langle -1, 1\rangle$.

Answer. The thought m is $\frac{1}{3}$.

Final Round of the 65th Czech and Slovak Mathematical Olympiad (April 4–5, 2016)

1. *Let $p > 3$ be a prime. Find the number of ordered sextuples (a, b, c, d, e, f) of positive integers, whose sum is $3p$, and all the fractions*

$$\frac{a+b}{c+d}, \quad \frac{b+c}{d+e}, \quad \frac{c+d}{e+f}, \quad \frac{d+e}{f+a}, \quad \frac{e+f}{a+b}$$

are integers. (Jaromír Šimša, Jaroslav Švrček)

Solution. Taking the product of the 1st, the 3rd and the 5th fractions reveals that their value has to be 1, that is

$$a+b = c+d = e+f = p. \tag{1}$$

the form of the second and of the fourth fraction implies

$$f+a \mid d+e \quad \text{and} \quad d+e \mid b+c. \tag{2}$$

that is first $f + a$ is at most the arithmetic mean of its multiples,

$$f+a \leqslant \tfrac{1}{3}((f+a)+(d+e)+(b+c)) = p, \tag{3}$$

and

$$f+a \mid (f+a)+(d+e)+(b+c) = 3p.$$

Thus $f + a$ divides $3p$ and is in the interval $\langle 2, p\rangle$. Consequently either $f + a = p$ or $f + a = 3$. We deal separately with these cases.

(i) Let $f + a = p$. Because of (3) there is $f + a = d + e = b + c = p$, which together with (1) gives $p - 1$ solutions of the form

$$(a, b, c, d, e, f) = (a, p-a, a, p-a, a, p-a), \quad \text{where } a \in \{1, 2, \dots, p-1\}.$$

(ii) Let $f + a = 3$. Then $\{a, f\} = \{1, 2\}$.

Firstly let $a = 1$ and $f = 2$. According to (1) then $b = p - 1$ and $e = p - 2$, and (2) has the form

$$3 \mid d + (p-2) \quad \text{and} \quad d + (p-2) \mid (p-1) + c. \tag{4}$$

In analyzing (4) we distinguish between $d = 1$ and $d \geqslant 2$.

If $d = 1$ then $c = p - 1$ and (4) reads

$$3 \mid p - 1 \quad \text{a} \quad p - 1 \mid 2(p - 1).$$

While the right relation always holds, the left one holds only for $p = 3q + 1$ (q is a suitable positive integer). For such prime numbers we get considering (1) solutions

$$(a, b, c, d, e, f) = (1, p - 1, p - 1, 1, p - 2, 2).$$

If $d \geqslant 2$ we show first, that the right relation in (4) is satisfied if and only if $d + (p - 2) = (p - 1) + c$ or $d = c + 1$. $d \geqslant 2$ namely implies $c = p - d \leqslant p - 2$, thus

$$d + (p - 2) \geqslant p \quad \text{and} \quad (p - 1) + c \leqslant 2p - 3 < 2p$$

and $d + (p - 2) = (p - 1) + c$. From $c + d = p$ and $d = c + 1$ we get $c = \frac{1}{2}(p - 1)$ a $d = \frac{1}{2}(p + 1)$. Since $d + (p - 2) = \frac{3}{2}(p - 1)$, the left relation in (4) is fulfilled and

$$(a, b, c, d, e, f) = (1, p - 1, \tfrac{1}{2}(p - 1), \tfrac{1}{2}(p + 1), p - 2, 2).$$

is a solution.

Finally $a = 2$ and $f = 1$. In this case $b = p - 2$ a $e = p - 1$, and (2) reads

$$3 \mid d + (p - 1) \quad \text{and} \quad d + (p - 1) \mid (p - 2) + c. \tag{5}$$

Because

$$d + (p - 1) \geqslant p \quad \text{and} \quad (p - 2) + c < 2p,$$

the right relation in (5) holds if and only if $d + (p - 1) = (p - 2) + c$, that is iff $c = d + 1$. Together with $c + d = p$ we get $c = \frac{1}{2}(p + 1)$ and $d = \frac{1}{2}(p - 1)$, thus the right relation in (5) holds as well, and the last solution is

$$(a, b, c, d, e, f) = (2, p - 2, \tfrac{1}{2}(p + 1), \tfrac{1}{2}(p - 1), p - 1, 1).$$

Conclusion. All the solutions found are apparently mutually different and their number depends on p modulo 3 ($p > 3$): If $p = 3q + 1$ then there are $p + 2$ sextuples, if $p = 3q + 2$, there are $p + 1$ sextuples.

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2. *Let r and r_a be the radii of inscribed circle and excircle opposite A of the triangle ABC. Show, that if*

$$r + r_a = |BC|,$$

then the triangle is right-angled. (Michal Rolínek)

Solution. Let us use the standard notation of the inner angles of the triangle ABC, further let I be the incenter and I_a be the excenter (of the excircle opposite A) and let D and E be in order the touching points of the thought circles. Since the bisectors BI and BI_a of the supplementary angles are perpendicular to each other (as well as CI and CI_a), the points B, C, I a I_a lie on the circle with the diameter II_a.

Thus D and E, the orthogonal projections of I and I_a onto the secant BC, are point reflections of each other with respect to the center of BC.

The right triangles BID and I_aBE are obviously similar and

$$|BD| : |ID| = |I_aE| : |BE| \quad \text{or} \quad |BD| \cdot |BE| = |ID| \cdot |I_aE|$$

considering the mentioned point reflection also

$$|BD| + |BE| = |BD| + |CD| = |BC| = r + r_a = |ID| + |I_aE|.$$

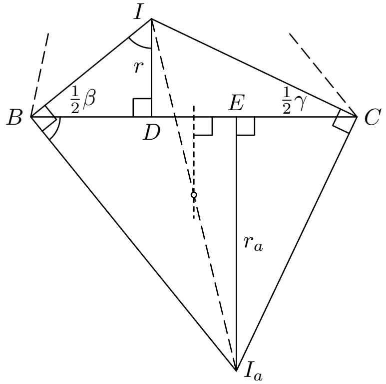

Fig. 1

The two equations imply that a pairs $(|ID|, |EI_a|)$ and $(|BD|, |BE|)$ are roots of the same quadratic equation, that is $|ID| = |BD|$ or $|ID| = |BE|$.

$|ID| = |BD|$ means the right-angled triangle BID is isosceles, which is $\beta = 90°$. Similarly, if $|ID| = |BE|$ that is $|ID| = |CD|$ (D and E are point reflections in the mentioned reflection) means the right triangle CID is isosceles, that is $\gamma = 90°$.

In both cases the triangle ABC is right-angled.

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3. *Mathematics clubs are very popular in certain city. Any two of them have at least one common member. Prove, that one can distribute rulers and compasses to the citizens in such a way that only one citizen gets both (compass and ruler) and any club has to his disposal both, compass and ruler, from its members.*

(Josef Tkadlec)

Solution. Let us consider the club K with the least number of its members (in case there is more such clubs, we take any). We give to one of it's members (let us call him Jacob) both, a compass and a ruler. Each of the other members of the club will get a compass. Any other citizen will get a ruler. We show, that this distribution comply with the conditions of the problem: Any club, which has Jacob as its member, has certainly both instruments.

If there is a club, where Jacob does not belong, then it has at least one common member with the club K, that is there is at least a compass at disposal in the club. If there were no ruler in the club, it would mean that it is a “subclub” of K and therefore has at least one member (Jacob) less than K, which is a contradiction with the choice of K. The described distribution really satisfies the conditions of the problem.

4. *For positive a, b, c it holds*

$$(a+c)(b^2+ac)=4a.$$

Find the maximal possible value of $b+c$ and find all triples (a,b,c), for which the value is attained. (Michal Rolínek)

Solution. We use the well know inequality $a^2+b^2 \geqslant 2ab$ to adjust the given one:

$$4a=(a+c)(b^2+ac)=a(b^2+c^2)+c(a^2+b^2)\geqslant a(b^2+c^2)+2abc=a(b+c)^2.$$

We can see, that $b+c \leqslant 2$, and also that the equality holds if and only if $0<a=b<2$ a $c=2-b>0$. Thats it.

5. *There is $|BC|=1$ in a triangle ABC and there is a unique point D on BC such that $|DA|^2=|DB|\cdot|DC|$. Find all possible values of the perimeter of ABC.* (Patrik Bak)

Solution. Let us denote by E the second intersection of AD with the circumcircle k. The power of D with respect to k gives $|DB|\cdot|DC|=|DA|\cdot|DE|$, which together with the given condition $|DA|^2=|DB|\cdot|DC|$ yields $|DA|=|DE|$. That is E lie the image p of the line BC in the homothety with center A and a coefficient 2 (Fig. 1).

Vice versa, to any intersection of a line p with the circle k we reconstruct the point D on BC, which fulfills $|DA|^2=|DB|\cdot|DC|$.

If the reconstruction have to be unique, the line p has to touch p in E.

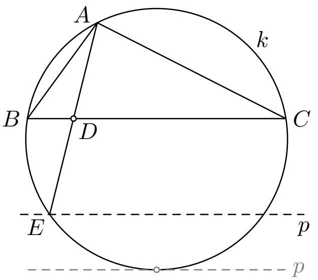

Fig. 2

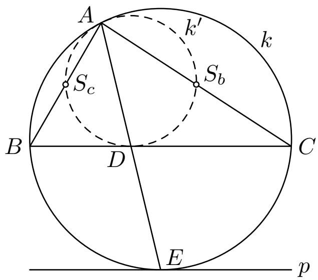

Fig. 3

Let us denote S_b and S_c in order the centers of AC and AB. The homothety with the center A and a coefficient $\frac{1}{2}$ sends A, B, C, E (lying on the circle k) to A, S_c, S_b, D which lie on the circle k' (Fig. 2), while the image of p is the tangent BC of k' in D. The powers of B and C with respect to k' give $|BD|^2=|BA|\cdot|BS_c|=\frac{1}{2}|BA|^2$ and $|CD|^2=|CA|\cdot|CS_b|=\frac{1}{2}|CA|^2$. All together for the perimeter of ABC:

$$|BC|+|AB|+|AC|=|BC|+\sqrt{2}(|BD|+|CD|)=|BC|+\sqrt{2}\cdot|BC|=1+\sqrt{2},$$

which is the only possible value.

6. *There is a figure of prince on a field of a 6×6 square chessboard. The prince can in one move jump either horizontally or vertically. The lengths of the jumps are alternately either one or two fields, and the jump on the next field is the first one. Decide, whether one can chose the initial field for the prince, so that the prince visits in an appropriate sequence of 35 jumps every field of the chessboard.*
(Peter Novotný)

Solution. Let us suppose, the appropriate sequence exists and let us enumerate the fields of the chessboard as follows:

1	2	3	4	1	2
2	3	4	1	2	3
3	4	1	2	3	4
4	1	2	3	4	1
1	2	3	4	1	2
2	3	4	1	2	3

The length one moves go from odd to even number and vice versa. The length two moves go from even to a different even number or from odd to a different odd number. If we denote $P_1, P_2, \dots, P_{36}$ the numbers of visited fields, then it follows, that among P_2, P_3, P_4, P_5 is each number (from 1 to 4) exactly once (P_2 and P_3 are different numbers with the same parity, and P_4, P_5 as well, only the parity is different). from the same reasons is any of the four numbers among P_{4k+2}, P_{4k+3}, P_{4k+4}, P_{4k+5} for arbitrary $k \in \{0, 1, \dots, 7\}$. Between the numbers $P_2, P_3, \dots, P_{33}$ is thus any of the numbers 1 to 4 exactly eight times.

The number 4 is on the chessboard just eight times, thus no from P_1, P_{34}, P_{35}, P_{36} can be 4. The numbers P_{34} and P_{35} have the same parity and are different (they are the length two move apart) The number 4 is not among them, therefore both must be odd. Then P_{36} has to be even and P_1 as well. Thus it has to be number 2.

The initial field (P_1) thus has to be one of the coloured fields on the left chessboard. One can repeat the arguments for the numbering of the right chessboard (just a rotation of the left one). Since no field has number 2 on both chessboard, we came to the contradiction. The initial field cannot be chosen.

1	**2**	3	4	1	**2**
2	3	4	1	**2**	3
3	4	1	**2**	3	4
4	1	**2**	3	4	1
1	**2**	3	4	1	**2**
2	3	4	1	**2**	3

2	3	4	1	**2**	3
1	**2**	3	4	1	**2**
4	1	**2**	3	4	1
3	4	1	**2**	3	4
2	3	4	1	**2**	3
1	**2**	3	4	1	**2**