Czech-Polish-Slovak Match

26 – 28 August 2020

(First day – 26 August 2020)

1. Let $ABCD$ be a parallelogram whose diagonals meet at P. Denote by M the midpoint of AB. Let Q be a point such that QA is tangent to the circumcircle of MAD and QB is tangent to the circumcircle of MBC. Prove that points Q, M, P are collinear. (*Patrik Bak, Slovakia*)

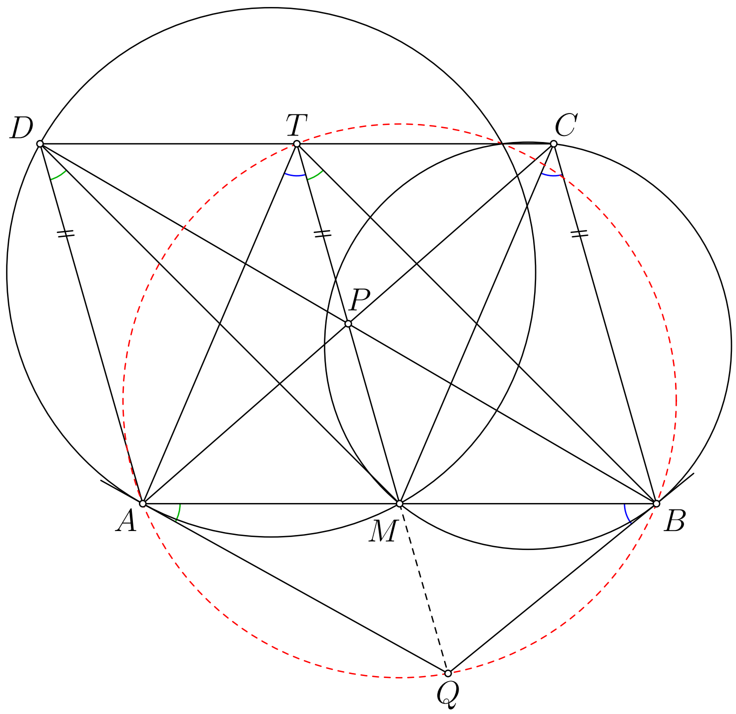
Solution 1. Let T be the midpoint of CD. Clearly T, M, P lie on a line parallel to BC and AD. Therefore

$$\angle ATB = \angle ATM + \angle MTB = \angle MCB + \angle ADM = \angle MBQ + \angle QAM = 180^\circ - \angle AQB,$$

which shows that A, T, B, Q are concyclic. Furthermore,

$$\angle ATQ = \angle ABQ = \angle MCB = \angle ATM,$$

which shows that T, M, Q are collinear.



Solution 2. Let T be the midpoint of CD and R be the reflection of M in T. Note that then triangles RTD and CBM are similar and hence

$$\angle MRD = \angle TRD = \angle BCM = \angle MBQ = \angle ABQ$$

and

$$\angle RMD = \angle ADM = \angle MAQ = \angle BAQ,$$

therefore triangles MRD and ABQ are similar. Note that the the segments QM and DT are medians of those triangles and vertices Q and D correspond to each other, hence triangles AMQ and MTD are similar. In particular, $\angle AMQ = \angle MTD = 180° - \angle AMT$, hence T, M, Q are colinear.

2. Given a positive integer n, we say that a real number x is n-good if there exist n positive integers $a_1, \ldots, a_n$ such that

$$x = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

Find all positive integers k for which the following assertion is true:

> If a, b are real numbers such that the closed interval $[a, b]$ contains infinitely many 2020-good numbers, then the interval $[a, b]$ contains at least one k-good number.

(*Josef Tkadlec, Czech Republic*)

Solution. We claim the answer is: All $k \geq 2019$.

First, we show that no $k \leq 2018$ belongs to the desired set. Consider the interval $I = [2019, 2020]$. The numbers $x_i = 1 + \cdots + 1 + 1/i$ (with number 1 appearing 2019 times) are all 2020-good and belong to I. On the other hand, any k-good number x with $k \leq 2018$ satisfies $x \leq 2018 \cdot 1/1 < 2019$. So no $k \leq 2018$ belongs to the desired set.

Note that the following Claim holds: For any $a < b \in \mathbb{R}$ and any $k \in \mathbb{N}$, if the half-open interval $[a, b)$ contains a k-good number then it contains an l-good number for any $l \geq k$. Indeed, this is immediate by induction (just take n large enough that adding a term $1/n$ does not reach or exceed b).

The Claim immediately implies that all $k \geq 2020$ belong to the desired set. Indeed, take two 2020-good numbers in $[a, b]$, say $x < y$. Then $x < b$, hence we are done by the Claim.

This only leaves $k = 2019$ unaccounted for. We show that it belongs to the set. Take an infinite sequence $X_0 = x_1, x_2, \ldots$ of 2020-good numbers in $[a, b]$ and consider their representations $x_i = 1/a_{i,1} + \cdots + 1/a_{i,2020}$, where without loss of generality, we have $a_{i,1} \leq \cdots \leq a_{i,2020}$, for each $i = 1, 2, \ldots$

Consider the set $S_1 = \{a_{i,1} | i = 1, 2, \ldots\}$ of the first (largest) terms of the representations. If it is finite, then some number b_1 is used as $a_{i,1}$ for infinitely many x_i. Focus on the (still infinite) subsequence $X_1 = \{x_i | a_{i,1} = b_1\}$ and consider the set $S_2 = \{a_{i,2} | x_i \in X_1\}$. If it is finite, proceed as before, restricting to an infinite set X_2 with fixed first two terms. At some point, the set S_j constructed in this way will be infinite (at the latest for $j = 2020$ after we have found infinitely many sequences agreeing on the first $j - 1 = 2019$ terms). At this point, consider $B = 1/b_1 + \cdots + 1/b_{j-1}$ (if $j = 1$, the sum is empty and we set $B = 0$) and take infinitely many $y_1, y_2, \ldots \in X_{j-1}$ whose representations agree on the first $j-1$ terms $1/b_1, \ldots, 1/b_{j-1}$ and whose j-th terms are pairwise different, say $c_1 < c_2 < \ldots$ Then

$y_i \le B + (2020 - j) \cdot 1/c_i \le B + 2020/i$. Since $y_i \ge a$ for each $i \in \mathbb{N}$, we have $B \ge a$. Moreover, $B \le y_1 \le b$, hence B is a 2019-good number contained in $[a, b]$.

3. The numbers $1, 2, \ldots, 2020$ are written on the blackboard. Venus and Serena play the following game. First, Venus connects by a line segment two numbers such that one of them divides the other. Then Serena connects by a line segment two numbers which has not been connected and such that one of them divides the other. Then Venus again and they continue until there is a triangle with one vertex in 2020, i.e. 2020 is connected to two numbers that are connected with each other. The girl that has drawn the last line segment (completed the triangle) is the winner. Which of the girls has a winning strategy? (*Tomáš Bárta, Czech Republic*)

Solution. We show that Venus has a winning strategy. Let us call a pair (k, l) *admissible* if k, $l \le 2019$ and (k, l) can be connected without causing immediate win of the rival. It means, k divides l or vice versa, (k, l) has not been connected yet and if both k and l are divisors of 2020, then neither of $(k, 2020)$, $(l, 2020)$ has been connected.

The idea is as follows: we show that there is an odd number of admissible pairs in the beginning and this number decreases by an odd number in each move with the only exceptions of moves $(2020, 1)$, $(2020, 4)$, and $(2020, 505)$. Therefore, Venus must guarantee that none or two of these three moves are played in the game. Then she has in each turn at least one admissible pair to play (odd number of them) and she cannot loose.

So, the winning strategy for Venus is as follows: In her first move Venus plays $(1, 2)$ (so, $(2020, 1)$ cannot be played by Serena). In all further turns Venus follows the following rules (in this order of priority) which implies that none or both of the pairs $(2020, 4)$, $(2020, 505)$ will be played:

1. if Serena draws second edge of a triangle, Venus completes the triangle and wins,
2. if Serena plays $(2020, 4)$, Venus plays $(2020, 505)$ and vice versa (then both these pairs will be played),
3. if Serena plays for the first time a pair from P_4 or P_{505}, Venus plays a pair from the other of the two sets, where

   $$P_S = \{(S, k), (2020, k) : k \text{ is a divisor of 2020 and a divisor or multiple of } S\},$$

   (then none of the pairs $(2020, 4)$, $(2020, 505)$ can be played any more),
4. otherwise, Venus plays any admissible pair. If none of the pairs from P_4, P_{505} is connected, Venus plays any admissible pair not belonging to these sets.

It is easy to see that Venus always can play according to points 2. and 3. It remains to show that she can follow the point 4. as well, i.e., that there are some admissible pairs (resp. admissible pairs not belonging to P_4, P_{505}) left. To do this let us focus on parity of admissible pairs now.

Let us first observe that if n has a prime factorization $p_1^{a_1} \dots p_k^{a_k}$ (with p_i being pairwise distinct), then n has $(a_1 + 1) \dots (a_k + 1)$ divisors and this number is odd if and only if all a_i's are even, i.e. n is a perfect square. Let us denote by $d(n)$ the number of positive divisors of n that are strictly less than n (call them proper divisors). We have: n is a perfect square if and only if $d(n)$ is even.

It is easy to see that total number of pairs that can be connected is

$$\sum_{n=1}^{2020} d(n),$$

which is an even number since there are 44 perfect squares less or equal to 2020 and therefore $2020 - 44$ of the numbers $d(n)$ are odd. Since $2020 = 2^2 \cdot 5 \cdot 101$ has 11 proper divisors, 11 of these pairs contain 2020, the remaining odd number of pairs are admissible.

If an admissible pair is connected, then number of admissible pairs decreases by 1. However, if a pair $(2020, S)$ is connected, then all pairs of type (S, k) belonging P_S are no more admissible. We compute the number of such pairs. If $S = 2^{a_1} 5^{a_2} 101^{a_3}$, $a_1 \in \{0, 1, 2\}$, a_2, $a_3 \in \{0, 1\}$, then the number of proper divisors of S is $(a_1+1)(a_2+1)(a_3 + 1) - 1$ and number of multiples of S larger than S that are proper divisors of 2020 is $(2 - a_1 + 1)(1 - a_2 + 1)(1 - a_3 + 1) - 2$. So, together we have

$$(a_1 + 1)(a_2 + 1)(a_3 + 1) + (3 - a_1)(2 - a_2)(2 - a_3) - 3 =: B$$

pairs which are no more admissible. Exactly one of the numbers $a_2 + 1$, $2 - a_2$ is even, exactly one of the numbers $a_3 + 1$, $2 - a_3$ is even and the numbers $a_1 + 1$, $3 - a_1$ have the same parity. Therefore B is even if and only if $a_1 \neq 1$ and $a_2 = a_3$, which means $k \in \{1, 4, 5 \cdot 101, 4 \cdot 5 \cdot 101\}$ (but the last number is equal 2020). So, we have shown that the number of admissible pairs decreases by an odd number in each move except $(2020, 1)$, $(2020, 4)$, $(2020, 505)$. Since, as we have just shown, number of admissible pairs in P_4 and P_{505} is even, there is always an odd number of admissible pairs not belonging to these sets. Therefore, there is always an admissible pair which can be played by Venus in the point 4. of the strategy. This completes the proof.

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26 – 28 August 2020

(Second day – 27 August 2020)

4. Let α be a given real number. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(x+y)(f(x)-f(y)) = \alpha(x-y)f(x+y)$$

holds for all x, $y \in \mathbb{R}$. (*Walther Janous, Austria*)

Solution. Consider first $\alpha = 0$. By setting $y = 0$ and $x \neq 0$, we infer $x(f(x) - f(0)) = 0$, that is $f(x) = f(0)$ for all real x. In the case $\alpha = 0$, all solutions are hence given by the constant functions. Henceforth, we can assume $\alpha \neq 0$. Plugging in $y = -x$, we get $0 = 2\alpha f(0) \implies f(0) = 0$. If there exists some $z \in \mathbb{R}$ with $f(z) \neq 0$, we set $x = z$ and $y = 0$ to obtain $zf(z) = \alpha z f(z)$. By assumption, this forces $\alpha = 1$. Hence, for $\alpha \in \mathbb{R} \setminus \{0, 1\}$ the only solution is the zero function. We are only left with $\alpha = 1$. Let $z \in \mathbb{R}$. For $x = z$ and $y = 1$ we get

$$(z+1)f(z) - (z+1)f(1) = (z-1)f(1+z),$$

whereas $x = z + 1$ and $y = -1$ leads to

$$zf(z+1) - zf(-1) = (z+2)f(z).$$

We can eliminate $f(z+1)$ from these two equations by multiplying the first one with z and the second one with $(z-1)$. Then, addition yields

$$(z+1)zf(z) - (z+1)zf(1) = (z-1)zf(-1) + (z-1)(z+2)f(z)$$

or equivalently

$$2f(z) = z(z+1)f(1) + z(z-1)f(-1),$$

and therefore $f(z) = az^2 + bz$ for some a, $b \in \mathbb{R}$. Checking in the original equation, we see that these are indeed solutions for all a, $b \in \mathbb{R}$.

In summary:

- For $\alpha = 0$ every constant function is a solution.
- For $\alpha = 1$ all functions $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) = ax^2 + bx$ and a, $b \in \mathbb{R}$ are solutions.
- In all other cases, the zero function is the only solution.

5. Let n be a positive integer and let $d(n)$ denote the number of ordered pairs of positive integers (x, y) such that

$$(x+1)^2 - xy(2x - xy + 2y) + (y+1)^2 = n.$$

Find the smallest positive integer n satisfying $d(n) = 61$. (*Patrik Bak, Slovakia*)

Solution. First, we modify the examined expression:

$$\begin{aligned}(x+1)^2 - xy(2x - xy + 2y) + (y+1)^2 &= \\ (x^2 + 2x + 1) - 2x^2y + x^2y^2 - 2xy^2 + (y+1)^2 &= \\ x^2(y^2 - 2y + 1) - 2x(y^2 - 1) + (y+1)^2 + 1 &= \\ (x(y-1) - (y+1))^2 + 1 &\end{aligned}$$

We see that we want to find the smallest integer n of the form $t^2 + 1$ such that the equation

$$x(y-1) - (y+1) = \pm t \tag{1}$$

has exactly 61 solutions over positive integers.

We may assume $t \geq 0$. For $y = 1$ we have $x(y-1) - (y+1) = -2$, for all x. So for $t = 2$ we have infinitely many solutions.

Assume that $y = 1$ yields no solution of (1). Then

$$x = \frac{\pm t + y + 1}{y - 1} = 1 + \frac{\pm t + 2}{y - 1}.$$

Clearly for $t = 0$ the solutions (x, y) are $(1, 3)$, $(1, 3)$ and for $t = 1$ the solutions are $(2, 4)$, $(4, 2)$, $(2, 2)$. Assume that $t \geq 3$. Then the negative sign before t cannot yield a solution, since

$$1 + \frac{-t + 2}{y - 1} \leq 1 + \frac{-3 + 2}{y - 1} = 1 - \frac{1}{y - 1} < 1.$$

Therefore it must hold

$$x = 1 + \frac{t + 2}{y - 1}.$$

The number of solutions of this equation is clearly equal to the number of divisors of $t + 2$. So we want to find t such that the number of divisors of $t + 2$ is exactly 61. Since 61 is a prime, $t + 2$ must be of form p^{60}, where p is a prime. Since we want to minimize n, we want to minimize t, therefore take $p = 2$. The minimal t is therefore $2^{60} - 2$, which gives us $n = (2^{60} - 2)^2 + 1$.

6. Let ABC be an acute triangle. Let P be a point such that PB and PC are tangent to circumcircle of ABC. Let X and Y be variable points on AB and AC, respectively, such that $\angle XPY = 2\angle BAC$ and P lies in the interior of triangle AXY. Let Z be the reflection of A across XY. Prove that the circumcircle of XYZ passes through a fixed point. (*Dominik Burek, Poland*)

Solution 1. Let BC intersect the circumcircle of CPY again at K. Let s be the spiral similarity centered at a point T such that $s(P) = X$ and $s(K) = Y$. Denote $Q = s(C)$. Then $\triangle CPK \sim \triangle QXY$, so

$$\angle XQY = \angle PCK = 180^\circ - \angle BCP = 180^\circ - \angle BAC = 180^\circ - \angle YZX.$$

Hence the circumcircle of XYZ passes through Q. To finish the solution, it is enough to prove that the position of Q does not depend on the choice of X and Y. In fact, we shall prove that $BACQ$ is a parallelogram.

Note that $\angle YCK = \angle ACB = \angle XBP$ and

$$\begin{aligned}\angle CKY &= 180^\circ - \angle YPC = 180^\circ - (360^\circ - \angle XPY - \angle CPB - \angle BPX) = \\ &= 180^\circ - 360^\circ + \angle XPY + \angle CPB + \angle BPX = \\ &= 180^\circ - 360^\circ + 2\angle BAC + (180^\circ - 2\angle BAC) + \angle BPX = \\ &= \angle BPX.\end{aligned}$$

Hence $\triangle KCY \sim \triangle PBX$.

Since $s(K) = Y$ and $s(P) = X$, there exists a spiral similarity s' centered at T such that $s'(K) = P$ and $s'(Y) = X$. Since $\triangle KCY \sim \triangle PBX$, we have $s'(C) = B$.

Let s'' be the spiral similarity centered at T such that $s''(P) = B$. Since $s'(C) = B$ and $s'(K) = P$, we have $s''(K) = C$. Since $s(K) = Y$ and $s(C) = Q$, we have $s''(Y) = Q$. It follows that $\triangle PKY \sim \triangle BCQ$.

So, $\angle QBC = \angle YPK = \angle YCK = \angle ACB$ and $\angle BCQ = \angle PKY = \angle PCY = \angle CBA$. It follows that $BACQ$ is a parallelogram. This finishes the proof.

Solution 2. Let Q be a point such that $BACQ$ is a parallelogram. We shall prove that the circumcircle of XYZ passes through Q. To that end, it is enough to prove $\angle XQY = 180^\circ - \angle BAC$ because then

$$\angle XQY + \angle YZX = (180^\circ - \angle BAC) + \angle BAC = 180^\circ.$$

Clearly,

$$\angle XBQ = \angle BAC = \angle PBC \quad \text{and} \quad \angle QCY = \angle BAC = \angle BCP. \tag{$\heartsuit$}$$

Note that

$$\angle CPB = 180^\circ - \angle PBC - \angle BCP = 180^\circ - 2\angle BAC.$$

Hence

$$\angle XPY + \angle CPB = 2\angle BAC + (180^\circ - 2\angle BAC) = 180^\circ.$$

It follows that P has isogonal conjugate P' in the quadrilateral $BCYX$. But ($\heartsuit$) implies that $P' = Q$.

So, P is the isogonal conjugate of Q in $BCYX$. Hence

$$\angle XQY + \angle CQB = 180^\circ.$$

Equivalently,

$$\angle XQY = 180^\circ - \angle CQB = 180^\circ - \angle BAC,$$

as required.