Czech-Polish-Slovak Match

IST Austria, 23 – 26 June 2019

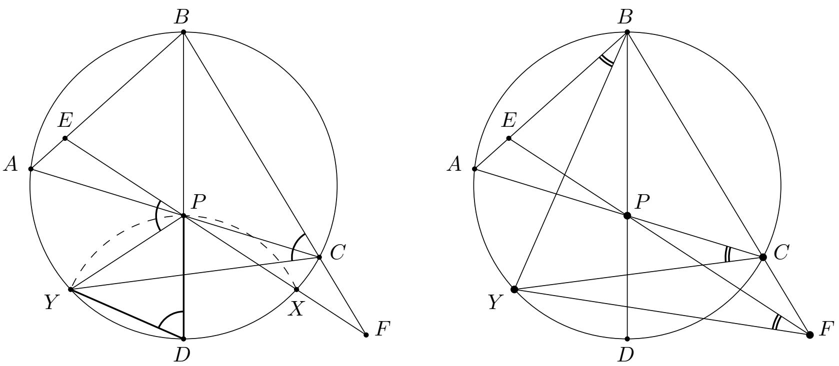
(First day – 24 June 2019)

1. Let ω be a circle. Points A, B, C, X, D, Y lie on ω in this order such that BD is its diameter and $DX = DY = DP$, where P is the intersection of AC and BD. Denote by E, F the intersections of line XP with lines AB, BC, respectively. Prove that points B, E, F, Y lie on a single circle. (*Patrik Bak, Slovakia*)

Solution. First, we show that the quadrilateral $YPCF$ is cyclic. Indeed, by simple angle-chasing we have

$$\angle YPF = 2 \cdot \angle YPD = 180^\circ - \angle BDY = 180^\circ - \angle BCY = \angle YCF.$$

The rest is angle-chasing again. We have $\angle EFY = \angle PFY = \angle PCY = \angle ACY = \angle ABY = \angle EBY$ as desired.



2. We consider positive integers n having at least six positive divisors. Let the positive divisors of n be arranged in a sequence $(d_i)_{1 \le i \le k}$ with

$$1 = d_1 < d_2 < \cdots < d_k = n \quad (k \ge 6).$$

Find all positive integers n such that

$$n = d_5^2 + d_6^2.$$

(*Walther Janous, Austria*)

Solution. In what follows we shall show that this question has the unique answer $n = 500$. Indeed, from $n = d_5^2 + d_6^2$ we readily infer that n has to be even. (For, otherwise d_5 and d_6 had to be odd. This in turn would yield n even.) Therefore $d_2 = 2$ is fixed. Furthermore from $d_5 \mid n$ we get $d_5 \mid d_6^2$ and similarly $d_6 \mid d_5^2$. This means:

Every prime dividing d_5 also divides d_6 and vice versa.

If d_5 has only one prime factor, i.e. it is a power of a prime, then $d_5 = p^k$ and $d_6 = p^{k+1}$. But since $p^k < 2p^k \le p^{k+1}$, it follows that $p = 2$ and $n = d_5^2 + d_6^2 = 2^{2k} + 2^{2k+2} = 5 \cdot 2^{2k}$. Therefore either $n = 20$, which is not a solution, or

$$d_2 = 2, \qquad d_3 = 4, \qquad d_4 = 5, \qquad d_5 = 8, \qquad d_6 = 10,$$

a contradiction.

Now d_5 and d_6 have at least two prime factors p and q with $p < q$ and $p^2q^2 \mid d_5^2 + d_6^2 = n$. Then $d_5 \ge pq$ and since $1 < p < q, p^2 < pq$ we also have $d_5 \le pq$. Now

$$d_2 = p = 2, \qquad \{d_3, d_4\} = \{q, p^2\} = \{q, 4\}, \qquad d_5 = pq = 2q, \qquad d_6 = p^2q = 4q.$$

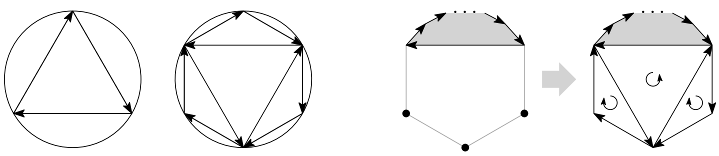
We get $n = d_5^2 + d_6^2 = 20q^2$, hence $q \le 5$. Checking the cases $q = 3$ and $q = 5$ gives the unique solution $n = 500$.

3. A dissection of a convex polygon into finitely many triangles by segments is called a *trilateration* if no three vertices of the created triangles lie on a single line (vertices of some triangles might lie inside the polygon). We say that a trilateration is *good* if its segments can be replaced with one-way arrows in such a way that the arrows along every triangle of the trilateration form a cycle and the arrows along the whole convex polygon also form a cycle. Find all $n \ge 3$ such that the regular n-gon has a good trilateration. (*Josef Greilhuber, Austria*)

Solution. We show that the regular n-gon has a good trilateration if and only if $3 \mid n$.

Given a regular n-gon and its good trilateration, color the triangles whose arrows go clockwise in black and the other ones in white. In this way, any two triangles sharing an edge have received different colors and all the triangles sharing an edge with the perimeter of the whole n-gon have received the same color (wlog black). We say that a segment in the trilateration is *interior* if it is not one of the sides of the n-gon. Let x be the number of interior segments. Since each interior segment is a side of precisely one white triangle and the sides of white triangle are all different interior segments, we have $3 \mid x$. Arguing likewise for the black triangles, we obtain $3 \mid x + n$. Hence $3 \mid n$.

It remains to show that when $3 \mid n$ then the regular n-gon has a good trilateration. This is straightforward by mathematical induction.



$n = 3$ $\qquad$ $n = 6$ $\qquad$ $n \to n + 3$

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4. Let α be a given real number. Determine all pairs (f, g) of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$xf(x+y) + \alpha \cdot yf(x-y) = g(x) + g(y)$$

for all $x, y \in \mathbb{R}$. (*Walther Janous, Austria*)

Solution. Depending on α, the solutions are given by:

- If $\alpha = 1$, then $f(x) = C$ and $g(x) = Cx$ for $x \in \mathbb{R}$ and C an arbitrary real constant.
- If $\alpha = -1$, then $f(x) = Cx$ and $g(x) = Cx^2$ for $x \in \mathbb{R}$ and C an arbitrary real constant.
- Else, $f(x) = g(x) = 0$ for $x \in \mathbb{R}$.

Letting $x = y = 0$, we obtain $2g(0) = 0$, thus $g(0) = 0$. Letting $y = 0$, we obtain $xf(x) = g(x)$ for all $x \in \mathbb{R}$. Thus, the equation can be rewritten as

$$xf(x+y) + \alpha yf(x-y) = xf(x) + yf(y). \tag{1}$$

Letting $x = 0$ in (1), we obtain $\alpha yf(-y) = yf(y)$. This yields

$$\forall x \neq 0\colon f(-x) = \alpha f(x). \tag{2}$$

If $f(x) = 0$ for all $x \neq 0$, we let $y = -x \neq 0$ in (1) and obtain $xf(0) = 0$, therefore f is the zero function, which always solves the equation.

Assume now that there exists $r \neq 0$ with $f(r) \neq 0$. Then it follows from (2) that $f(r) = \alpha f(-r) = \alpha^2 f(r)$, thus $\alpha^2 = 1$ and hence $\alpha \in \{\pm 1\}$.

The right-hand side of (1) is symmetric in x and y. By switching x and y, we thus obtain the equation

$$xf(x+y) + \alpha yf(x-y) = yf(x+y) + \alpha xf(y-x).$$

For $r \in \mathbb{R}$ we let $x = (r+1)/2$ and $y = (r-1)/2$, which yields

$$f(r) = \alpha \frac{r+1}{2} f(-1) - \alpha \frac{r-1}{2} f(1).$$

By (2), we obtain

$$f(r) = \frac{\alpha f(1)}{2} \Big(\alpha(r+1) - (r-1) \Big).$$

In the case $\alpha = 1$ this means $f(r) = f(1)$ for all $r \in \mathbb{R}$. In the case $\alpha = -1$ this means $f(r) = rf(1)$ for all $r \in \mathbb{R}$. Both functions solve the equation, as can be checked easily.

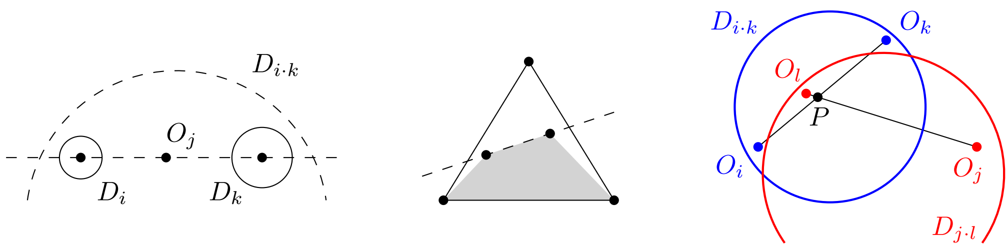
5. Determine whether there exist 100 disks $D_2, D_3, \ldots, D_{101}$ in the plane such that the following conditions hold for all pairs (a, b) of indices satisfying $2 \le a < b \le 101$:

1. If $a \mid b$ then D_a is contained in D_b.
2. If $\mathrm{GCD}(a, b) = 1$ then D_a and D_b are disjoint.

(A disk $D(O, r)$ is a set of points in the plane whose distance to a given point O is at most a given positive real number r.) (*Josef Greilhuber & Josef Tkadlec, Austria*)

Solution. Such disks do not exist. Suppose otherwise and denote by O_i the center of the disk D_i. Consider the set $S = \{O_2, O_3, O_5, O_7, O_{11}\}$ of centers of five disks with pairwise coprime indices. We distinguish two cases:

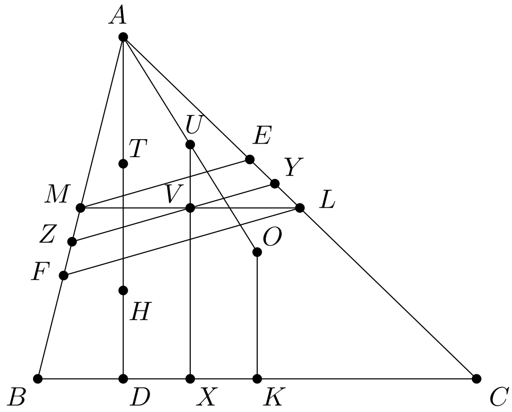
(i) Some three points from S lie on a single line: Suppose the three collinear points are O_i, O_j, O_k in this order. Then $i \cdot k \le 7 \cdot 11 \le 101$, hence the disk $D_{i \cdot k}$ is defined. By 1., it contains both D_i and D_k, thus it contains O_i and O_k and by convexity it also contains O_j. Therefore, disks D_j, $D_{i \cdot k}$ intersect, a contradiction with 2.



(ii) No three points from S lie on a single line: Then there exist four points from S that form a convex quadrilateral. (Indeed, either the convex hull of S contains at least four points, or it is a triangle. In the latter case, the line passing through the two interior points intersects two sides of the triangle and the two interior points form a convex quadrilateral with the endpoints of the side that is not intersected.) Suppose the four vertices of the convex quadrilateral are O_i, O_j, O_k, O_l in this order. Then, as before, both $i \cdot k$ and $j \cdot l$ are at most $7 \cdot 11 \le 101$ hence the disks $D_{i \cdot k}$ and $D_{j \cdot l}$ are defined. By 1. and by convexity, they both contain the intersection P of diagonals of $O_iO_jO_kO_l$, which is a contradiction with 2.

6. Let ABC be an acute triangle with $AB < AC$ and $\angle BAC = 60°$. Denote its altitudes by AD, BE, CF and its orthocenter by H. Let K, L, M be the midpoints of sides BC, CA, AB, respectively. Prove that the midpoints of segments AH, DK, EL, FM lie on a single circle. (*Dominik Burek, Poland*)

Solution. Denote the midpoints of AH, DK, EL, FM by T, X, Y, Z, respectively. Furthermore, let O be the circumcenter of triangle ABC and U the midpoint of AO



(that is, the circumcenter of triangle AML). We will show that U lies on the circle too.

First, we show that $TUYZ$ is cyclic. In fact, we show that is is an isosceles trapezoid whose line of symmetry is the angle bisector of $\angle BAC$: Since $\angle BAC = 60°$, we have $AE = \frac{1}{2}AB = AM$, thus $\triangle AME$ is equilateral and, likewise, $\triangle AFL$ is equilateral. Since Y and Z are the midpoints of lateral sides EL, MF of a trapezoid $ELFM$, triangle AYZ is also equilateral and the perpendicular bisector of YZ is the angle bisector of $\angle BAC$. Regarding TU, since lines AT and AU are isogonal in $\angle BAC$ and $AF = AL$, the right triangles AFH and ALO are congruent. Thus the perpendicular bisector of TU is the angle bisector of $\angle BAC$ as well.

Second, we show that $UYXZ$ is cyclic: Let V be the center of parallelogram $AMKL$. Since V is the midpoint of ML, it lies on the midline YZ of trapezoid $MELF$. Since it is the midpoint of AK, it also lies on the midline UX of trapezoid $AOKD$. Thus, it remains to check that $VY \cdot VZ = VU \cdot VX$, which is straightforward. For the left-hand side, we have $VY = \frac{1}{2}ME = \frac{1}{4}AB$ and $VZ = \frac{1}{2}LF = \frac{1}{2}AF$. For the right-hand side, we have $VU = \frac{1}{2}OK = \frac{1}{4}AH$ and $VX = \frac{1}{2}AD$. Plugging this in, we need $AB \cdot AF = AH \cdot AD$ which follows from $BFHD$ being cyclic.

Since both $TUYZ$ and $UYXZ$ are cyclic, so is $TYXZ$.