2018

67th Czech and Slovak Mathematical Olympiad

Translated into English by
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First Round of the 67th Czech and Slovak Mathematical Olympiad
Problems for the take-home part
(October 2017)

1. *Paul is filling the cells of a rectangular table alternately with crosses and circles (he starts with a cross). When the table is filled in completely, he determines his score as $O-X$ where O is the total number of rows and columns containing more circles than crosses and X is the total number of rows and columns containing more crosses than circles.*

a) Prove that for a $2 \times n$ table, the score is always equal to 0.

b) In terms of n, what is the largest possible score Paul can achieve for a $(2n+1) \times (2n+1)$ table? (Josef Tkadlec)

Solution. a) Consider a table with 2 rows and n columns filled in with n crosses and n circles. Since the total number of crosses and circles is the same, crosses dominate in one row if and only if circles dominate in the other one. Hence the rows contribute 0 to the total score.

Next, denote by x, e, and o the number of columns containing two, one, and zero crosses, respectively. Since the table contains a total of n crosses and n circles, we have $2x + e = n = e + 2o$, hence $x = o$. As x and o are the number of columns dominated by crosses and circles, respectively, the columns contribute 0 to the total score too.

b) Consider a $(2n+1) \times (2n+1)$ table filled with $\frac{1}{2}((2n+1)^2 - 1) = 2n(n+1)$ circles and $2n(n+1)+1$ crosses. Since $2n+1$ is odd, each row and column is dominated by one of the two symbols. Circles can dominate in at most $2n(n+1)/(n+1) = 2n$ rows and thus at least one row is dominated by crosses. Likewise for columns, hence $O \leqslant 2n + 2n = 4n$, $X \geqslant 1 + 1 = 2$ and therefore $O - X \leqslant 4n - 2$.

Finally, we argue that the score $4n-2$ can be achieved for any n. It suffices to specify a set $\mathcal{S}$ of $2n(n+1)$ cells that are to be filled with circles. An example is a set $\mathcal{S}$ that consists of $n+1$ "parallel diagonals" in the top-left $2n \times 2n$ subsquare of the table and no other cells in the bottom row or right column (see Fig. 1 for $n = 3$).

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2. *Let a, b be real numbers such that $a+b > 2$. Prove that the system of inequalities*

$$(a-1)x + b < x^2 < ax + (b-1)$$

has infinitely many real solutions x. (Jaromír Šimša)

Solution. We rewrite the system as

$$F(x) > 0 \wedge G(x) < 0,$$

O	×	×	O	O	O	×
O	O	×	×	O	O	×
O	O	O	×	×	O	×
O	O	O	O	×	×	×
×	O	O	O	O	×	×
×	×	O	O	O	O	×
×	×	×	×	×	×	×

Fig. 1

where $F(x) = x^2-(a-1)x-b$ and $G(x) = x^2-ax-b+1$. Observe that $F(x)-G(x) = x-1$.

The condition $a+b > 2$ implies that

$$F(1) = G(1) = 2 - a - b < 0,$$

hence $x = 1$ is not a solution. However, $G(1) < 0$ implies that the quadratic equation $G(x) = 0$ has a root $x_0 > 1$. Then

$$F(x_0) = F(x_0) - 0 = F(x_0) - G(x_0) = x_0 - 1 > 0.$$

From $F(1) < 0$ and $F(x_0) > 0$ we deduce that there exists a root x_1 of $F(x) = 0$ that belongs to the open interval $(1, x_0)$. Since

$$F(1) < 0 \wedge F(x_1) = 0, \quad \text{and} \quad G(1) < 0 \wedge G(x_0) = 0,$$

any $x \in (x_1, x_0)$ is a solution to the original system.

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3. *Two externally tangent unit circles are given in the plane. Consider any rectangle (or a square) containing both the circles such that each side of the rectangle is tangent to at least one circle. Find the largest and the smallest possible area of such a rectangle.* (Jaroslav Švrček)

Solution. Denote the circles by k_1, k_2, their radius by $r = 1$, and their centers by O_1, O_2, respectively. Let $ABCD$ be one such rectangle (or a square) and without loss of generality assume that the sides AB, BC are tangent to k_1 while the sides CD, DA are tangent to k_2. Let P be the intersection of a line through O_1 parallel to AB and a line through O_2 parallel to BC. Finally, let $\phi = \angle PO_1O_2$ ($\phi \in [0, \frac{1}{4}\pi]$, Fig. 2).

Then

$$[ABCD] = AB \cdot BC = (2r + 2r\cos\phi)(2r + 2r\sin\phi) = 4(1+\sin\phi)(1+\cos\phi).$$

It remains to analyze the expression $V(\phi) = (1+\sin\phi)(1+\cos\phi)$ for $\phi \in [0, \frac{1}{4}\pi]$. Multiplying out, this rewrites as

$$V(\phi) = 1 + \sin\phi + \cos\phi + \sin\phi\cos\phi = \tfrac{1}{2} + (\sin\phi + \cos\phi) + \tfrac{1}{2}(\sin\phi + \cos\phi)^2.$$

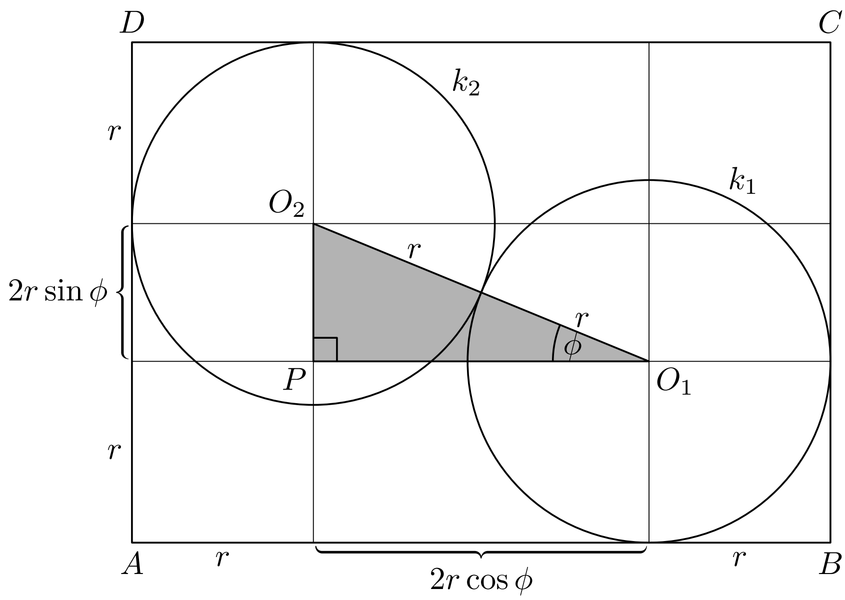

Fig. 2

and it remains to analyze $u = \sin\phi + \cos\phi$. Squaring and using the formula $\sin 2\phi = 2\sin\phi\cos\phi$, we obtain $1 \leqslant u \leqslant \sqrt{2}$. Since the function $V(\phi) = \frac{1}{2} + u + \frac{1}{2}u^2$ is increasing on interval $[1, \sqrt{2}]$, we get $2 \leqslant \phi \leqslant \frac{3}{2} + \sqrt{2}$ and finally

$$8 \leqslant [ABCD] \leqslant 6 + 4\sqrt{2}.$$

The first inequality is sharp for $\phi = 0$, that is if both AB and CD are tangent to both the circles. The second inequality is sharp for $\phi = \frac{1}{4}\pi$, that is if $ABCD$ is a square.

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4. *Find the largest positive integer n such that*

$$\lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \lfloor\sqrt{3}\rfloor + \cdots + \lfloor\sqrt{n}\rfloor$$

is a prime ($\lfloor x\rfloor$ denotes the largest integer not exceeding x). (Patrik Bak)

Solution. Consider the infinite sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \lfloor\sqrt{n}\rfloor$. This sequence is clearly non-decreasing and since

$$k = \sqrt{k^2} < \sqrt{k^2+1} < \cdots < \sqrt{k^2+2k} < \sqrt{k^2+2k+1} = k+1,$$

it contains every integer k precisely $(2k+1)$-times. This allows us to express the value of the sum $s_n = \sum_{i=1}^{n} a_i$ as follows: Let $k = \lfloor\sqrt{n}\rfloor$, that is $n = k^2 + l$ for some $l \in \{0, 1, \ldots, 2k\}$. Then

$$\begin{aligned} s_n &= \sum_{i=0}^{k-1} i(2i+1) + k\cdot(l+1) = 2\cdot\sum_{i=1}^{k-1} i^2 + \sum_{i=1}^{k-1} i + k(l+1) \\ &= 2\cdot\frac{(k-1)(k-1+1)(2(k-1)+1)}{6} + \frac{(k-1)(k-1+1)}{2} + k(l+1) \\ &= \frac{(k-1)k(4k+1)}{6} + k(l+1), \end{aligned}$$

where we used $1+2+\cdots+n=\frac{1}{2}n(n+1)$ and $1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

If $k>6$ then the fraction $\frac{1}{6}(k-1)k(4k+1)$ is an integer sharing a prime factor with k, hence the whole right-hand side is sharing a prime factor with $k<s_n$ and s_n is not a prime.

If $k \leqslant 6$ then $n<(6+1)^2=49$. Plugging $n=48$ into the right-hand side we get $s_{48}=203=7\cdot 29$. For $n=47$ we get $s_{47}=197$, which is a prime. The answer is $n=47$.

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5. *Let $ABCD$ be a convex quadrilateral such that $\angle ABC=\angle ACD$ and $\angle ACB=\angle ADC$. Suppose that the circumcenter O of triangle BCD is different from A. Prove that the angle OAC is right.* (Patrik Bak)

Solution. Since $\angle ABC+\angle CDA<180°$, point A lies inside the circumcircle ω of triangle BCD. Denote by C', D' the second intersection of ω with rays CA, DA, respectively (Fig. 3). We angle chase:

$$\angle D'C'C=\angle D'DC=\angle ADC=\angle ACB.$$

Hence $BCC'D'$ is an isosceles trapezoid. Moreover, since $\angle C'AD'=\angle CAD=\angle BAC$, triangles ABC and $AD'C'$ are similar by AA and in fact due to $BC=C'D'$ they are congruent. Point A is thus the midpoint of the chord CC' and $\angle OAC=90°$ follows.

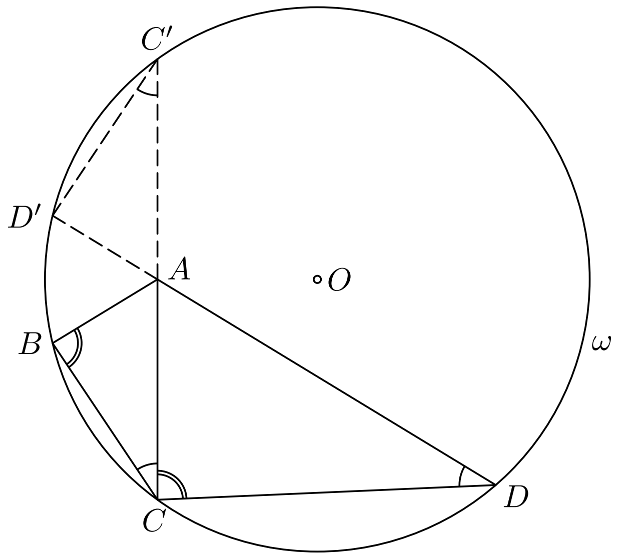

Fig. 3

Another solution. Let's frame the figure with respect to triangle ABD. Then AC is the A-angle bisector. The Inscribed angle theorem states that the (reflex) angle BOD is twice the (convex) angle BCD, hence for the size of the convex angle BOD we get

$$\angle BOD=360°-2\cdot\angle DCB=360°-\angle ADC-\angle BCD-\angle ABC=\angle BAD.$$

Therefore O lies on the arc BAD of the circumcircle of triangle ABD. Since $OB = OD$, point O is the midpoint of that arc and thus it lies on the external A-angle bisector which is perpendicular to the A-angle bisector.

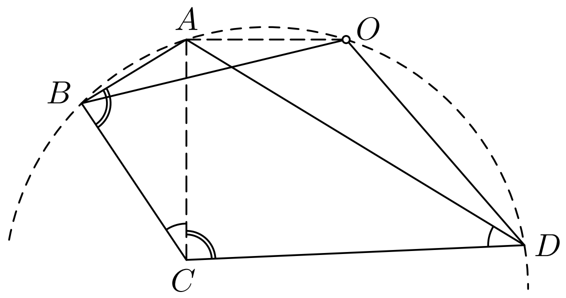

Fig. 4

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6. *Find the largest possible size of a set $\mathbb{M}$ of integers with the following property: Among any three distinct numbers from $\mathbb{M}$, there exist two numbers whose sum is a power of 2 with non-negative integer exponent.* (Ján Mazák)

Solution. The set $\{-1, 3, 5, -2, 6, 10\}$ attests that $\mathbb{M}$ can have 6 elements: The sum of any two numbers from the triplet $(-1, 3, 5)$ is a power of two and the same is true for triplet $(-2, 6, 10)$. For the sake of contradiction, assume that some set $\mathbb{M}$ has more than 6 elements.

Clearly, $\mathbb{M}$ can't contain three (or more) non-positive numbers. Hence it contains at least five positive numbers. Denote by x the largest positive number in $\mathbb{M}$ and by a, b, c, d some four other positive numbers in $\mathbb{M}$. Consider pairs $x + a$, $x + b$, $x + c$, $x + d$. They are all larger than x and less than $2x$. The open interval $(x, 2x)$ contains at most one power of two, hence at least three of the four sums are not a power of two. Without loss of generality, assume those are $x + a$, $x + b$, $x + c$. Considering the triplets (a, b, x), (a, c, x), (b, c, x) we infer that all $a + b$, $a + c$, $b + c$ are powers of two. However, this is impossible. Without loss of generality, let $a = \max\{a, b, c\}$. Then $a + b$ and $a + c$ both lie in $(a, 2a)$, hence at least one of them is not a power of two, a contradiction.

First Round of the 67th Czech and Slovak Mathematical Olympiad (December 12th, 2017)

1. *Find all real numbers p such that the system*

$$x^2 + (p-1)x + p \leqslant 0$$
$$x^2 - (p-1)x + p \leqslant 0$$

of inequalities has at least one solution $x \in \mathbb{R}$. (Jaromír Šimša)

Solution. If $p \leqslant 0$ then $x = 0$ is clearly a solution. If $p > 0$ then summing up we get $2x^2 + 2p \leqslant 0$ which doesn't hold for any real x.

Answer. The answer is $p \in (-\infty, 0]$.

Another solution. The graphs of functions $f(x) = x^2+ux+v$ and $g(x) = x^2-ux+v$ are symmetric about the y axis, hence the solutions to the inequalities $f(x) \leqslant 0$, $g(x) \leqslant 0$ are two (possibly degenerate) intervals symmetric about 0. The intersection of these intervals is nonempty if and only if $v = f(0) = g(0) \leqslant 0$. In our case, this happens if and only if $p \leqslant 0$.

2. *Let ABC be a triangle and S_b, S_c the midpoints of the sides AC, AB, respectively. Prove that if $AB < AC$ then $\angle BS_cC < \angle BS_bC$.* (Patrik Bak)

Solution. It suffices to prove that if $AB < AC$ then S_b lies inside the circumcircle k of triangle BS_cC.

The midline S_bS_c is parallel to BC (Fig. 1). Let line S_bS_c intersect k for the second time at P. We will show that S_b lies on the segment S_cP (as opposed to lying on the ray opposite to PS_c). To that end, it suffices to prove $\angle BCA < \angle BCP$. By symmetry about the perpendicular bisector of BC we have $\angle BCP = \angle CBA$, so we need to prove $\angle BCA < \angle CBA$ which is in fact clearly equivalent to the given $AB < AC$.

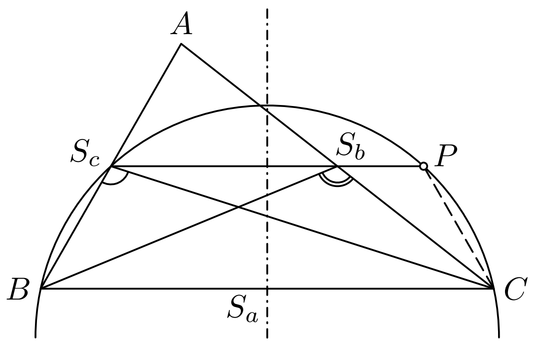

Fig. 1

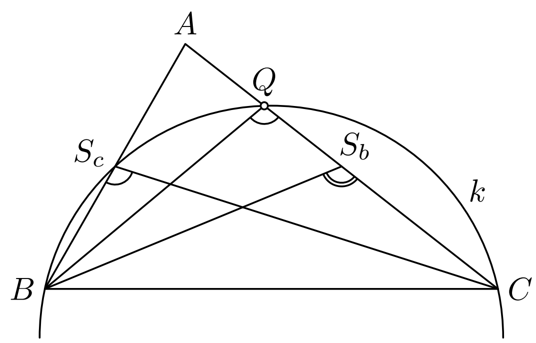

Fig. 2

Another solution. By power of A with respect to k, there exists a point Q on the ray AC such that $AQ \cdot AC = AB \cdot AS_c = \frac{1}{2}AB^2$. Then (Fig. 2)

$$AQ = \frac{AB^2}{2 \cdot AC} < \frac{AC^2}{2 \cdot AC} = \frac{1}{2}AC = AS_b,$$

hence Q lies on segment AS_b. As before we conclude that S_b lies inside the circumcircle of triangle BS_cC.

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3. *Paul is filling the cells of a rectangular table alternately with crosses and circles (he starts with a cross). When the table is filled in completely, he determines his score as $X + O$ where X is the number of rows containing more crosses than than circles and O is the number of columns containing more circles than crosses. In terms of n, what is the largest possible score Paul can achieve for a $(2n+1) \times (2n+1)$ table?* (Josef Tkadlec)

Solution. In total there are $2n(n+1)+1 < (2n+1)(n+1)$ crosses and $2n(n+1)$ circles. Hence the crosses can dominate in at most $2n$ rows and, similarly, circles can dominate in at most $2n$ columns for the total score $2n + 2n = 4n$.

Such a score can be achieved if, for example, Paul draws crosses in the left $n+1$ columns of the first n rows, the right $n+1$ columns of the last n rows and the middle cell of the middle row. That is precisely $2n(n+1)+1$ crosses and we easily check that crosses dominate in all rows except for the middle one while circles dominate in all columns except for the middle one.

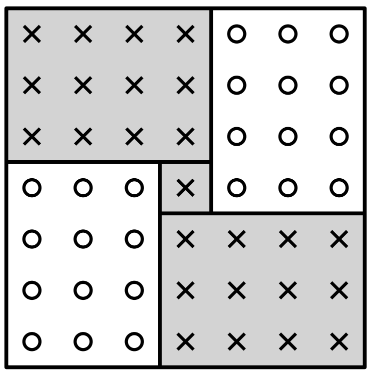

Fig. 3

Second Round of the 67th Czech and Slovak Mathematical Olympiad (January 16th, 2018)

1. *Paul is filling the cells of a rectangular table alternately with crosses and circles (he starts with a cross). When the table is filled in completely, he determines his score as $X - O$ where X is the sum of squares of the numbers of crosses in all the rows and columns, and O is the sum of squares of the numbers of circles in all the rows and columns. Find all possible values of the score for a 67×67 table.* (Josef Tkadlec)

Solution. Let $n = 67$ and denote by $k = \frac{1}{2}(n^2+1)$ the total number of crosses in the table. A row containing a crosses and $n-a$ circles contributes $a^2-(n-a)^2 = 2n\cdot a-n^2$ to the total score and thus all the n rows combined contribute

$$2n \cdot k - n \cdot n^2 = 2n \cdot \frac{n^2+1}{2} - n^3 = n$$

to the total score. Likewise, columns contribute n. Hence the total score is always equal to $2n = 134$.

Another solution. Consider an $n \times n$ table filled with arbitrarily many crosses and circles. We show that replacing any circle by a cross increases the score by $4n$. Since the score for a table filled with all circles equals $-2n^3$ and Paul's table contains $\frac{1}{2}(n^2+1)$ crosses, the final score will always be equal to $-2n^3 + 4n \cdot \frac{1}{2}(n^2+1) = 2n$.

Consider any cell containing a circle and denote by r and c the number of crosses in its row and column, respectively. The contribution of this row and column changes from

$$A = r^2 - (n-r)^2 + s^2 - (n-s)^2 = 2n(r+s) - 2n^2$$

to

$$B = (r+1)^2 - (n-r-1)^2 + (s+1)^2 - (n-s-1)^2 = 2n\,(r+1+s+1) - 2n^2$$

and the contribution of other rows and columns doesn't change. Since $B - A = 4n$, we are done.

2. *Let k be a semicircle with diameter PQ. Consider a chord BC of fixed length d whose endpoints are distinct from P, Q. A ray of light emanating from B reaches point C after reflecting from PQ at such a point A that $\angle PAB = \angle QAC$. Prove that $\angle BAC$ doesn't depend on the position of the chord BC on k.*

(Šárka Gergelitsová)

Solution. Reflect k and C about PQ to get l and C', respectively (Fig. 1). Then C' lies on l and since $\angle QAC' = \angle QAC = \angle PAB$ it also lies on BA. Triangle $C'CA$ is isosceles, hence

$$\angle BAC = \angle AC'C + \angle ACC' = 2 \cdot \angle BC'C$$

The chord BC of circle $k \cup l$ has a fixed length, hence the corresponding inscribed angle $BC'C$ has fixed size and we may conclude.

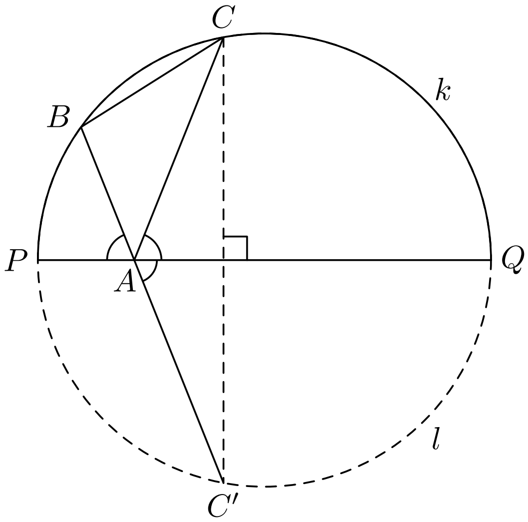

Fig. 1

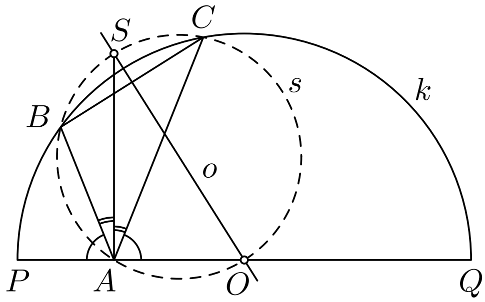

Fig. 2

Another solution. Let O be the midpoint of PQ. We will show that O lies on the circumcircle of triangle ABC (Fig. 2). This will imply that $\angle BAC = \angle BOC$ which is clearly fixed.

Observe that O lies on the perpendicular bisector of BC. Moreover, if $O \neq A$ then AO is the external A-angle bisector with respect to triangle ABC. Therefore O is the midpoint of arc BAC.

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3. *Let $a \neq b$ be positive real numbers. Consider the equation*

$$\lfloor ax + b \rfloor = \lfloor bx + a \rfloor$$

where $\lfloor y \rfloor$ denotes the largest integer not exceeding y. Prove that the set of real solutions x to this equation contains an interval of length at least

$$\frac{1}{\max\{a, b\}}.$$

(Patrik Bak)

Solution. Consider linear functions $f(x) = ax + b$, $g(x) = bx + a$. Since a, b are distinct and positive, their graphs are two distinct lines with positive slope. As $f(1) = g(1) = a + b$, point $P = [1, a + b]$ is the intersection of these lines (Fig. 3).

Without loss of generality, assume $b > a$ (i.e. the line determined by g is the "steeper" one). Then $f(x) > g(x)$ for $x < 1$, whereas $f(x) < g(x)$ for $x > 1$: indeed,

$$f(x) - g(x) = (ax + b) - (bx + a) = (b - a)(1 - x).$$

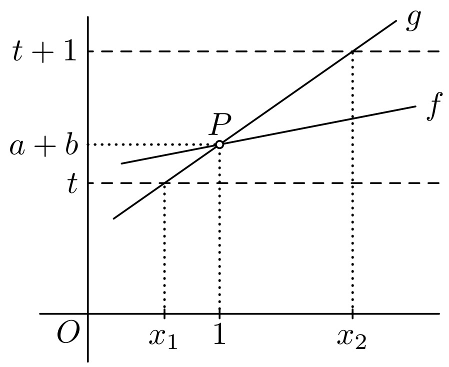

Fig. 3

Let $t = \lfloor a+b \rfloor$ and consider $x_1 \leqslant 1 < x_2$ such that $g(x_1) = t$ and $g(x_2) = t+1$ (that is, $x_1 = \frac{t-a}{b}$ and $x_2 = \frac{t+1-a}{b}$). We claim that the interval $[x_1, x_2)$ has all the desired properties.

First, for any $x \in [x_1, x_2)$ we have

$$t = g(x_1) \leqslant \min\{f(x), g(x)\} \leqslant \max\{f(x), g(x)\} < g(x_2) = t+1,$$

and thus x is a solution to the equation.

Second,

$$1 = (t+1) - t = bx_2 + a - (bx_1 + a) = b(x_2 - x_1),$$

and thus $x_2 - x_1 = 1/b = 1/\max\{a, b\}$ and the interval has the desired length.

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4. *Do there exist positive integers n, k such that*

$$\frac{n}{11^k - n}$$

is a square of an integer? (Ján Mazák)

Solution. Such numbers don't exist. For the sake of contradiction, assume that there exist positive integers n, k, a such that

$$\frac{n}{11^k - n} = a^2$$

which rewrites as

$$n(a^2 + 1) = a^2 \cdot 11^k.$$

From $\mathrm{GCD}(a^2, a^2+1) = 1$ we deduce $a^2+1 \mid 11^k$ and hence $a^2+1 = 11^t$ for $1 \leqslant t \leqslant k$. In particular, $a^2 \equiv 10 \pmod{11}$. However, this is impossible as the squares of integers give remainders $0, 1, 4, 9, 5, 3, 3, 5, 9, 4, 1, \ldots$ upon division by 11.

Final Round of the 67th Czech and Slovak Mathematical Olympiad (March 18–21, 2018)

1. *In a certain club, some pairs of members are friends. Given $k \geqslant 3$, we say that a club is k-good if every group of k members can be seated around a round table such that every two neighbors are friends. Prove that if a club is 6-good then it is 7-good.* (Josef Tkadlec)

Solution. Consider a 6-good club and denote some seven of its members by $A, \ldots, G$. It suffices to show that $A, \ldots, G$ can be seated around a table as required. Consider only friendships among $A, \ldots, G$. First, we show that every member has at least three friends.

Without loss of generality consider G. By assumption, $B, \ldots, G$ can be seated as required, hence G has at least two friends. Without loss of generality, F is one of them. By assumption, $A, \ldots, E, G$ (omitting F) can be seated as required, hence G has at least two more friends apart from F for a total of at least three friends.

Since every member has at least three friends, there exists a member with at least four friends (otherwise the number of friendly pairs equals $\frac{1}{2} \cdot 7 \cdot 3$, which is clearly impossible). Without loss of generality, assume G has at least four friends.

By assumption, $A, \ldots, F$ can be seated as required. In such a seating, some two of the four friends of G are neighbors and we can seat G in between them.

Remark. The statement "If a club is k-good then it is $(k+1)$-good" holds precisely for $k \in \{3, 4, 5, 6, 7, 8, 10, 11, 13, 16\}$. The counterexamples are called *hypohamiltonian graphs.* For $k = 9$, one such example is the *Petersen graph* (Fig. 1).

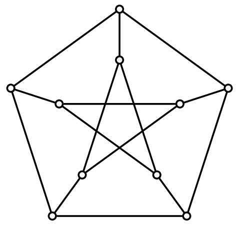

Fig. 1

2. *Let x, y, z be real numbers such that*

$$\frac{1}{|x^2 + 2yz|}, \quad \frac{1}{|y^2 + 2zx|}, \quad \frac{1}{|z^2 + 2xy|}$$

are side-lengths of a (non-degenerate) triangle. Find all possible values of $xy + yz + zx$. (Michal Rolínek)

Solution. If $x = y = z = t > 0$ then the three fractions are sides of an equilateral triangle and $xy + yz + zx = 3t^2$, hence $xy + yz + zx$ can attain all positive values. Similarly, for $x = y = t > 0$ and $z = -2t$ the three fractions are $\frac{1}{3}t^{-2}$, $\frac{1}{3}t^{-2}$, $\frac{1}{6}t^{-2}$ which are positive numbers that are side-lengths of an isosceles triangle ($\frac{1}{6} < \frac{1}{3} + \frac{1}{3}$). Since $xy + yz + zx = -3t^2$, any negative value can be attained too.

Next we show that $xy + yz + zx$ can't be 0. Assume otherwise. Numbers x, y, z are mutually distinct: if, say, x and y were equal then the denominator of the first fractions would be equal to $|x^2 + 2yz| = |xy + (yz + xz)| = 0$ which is impossible.

Let's look at the fractions without absolute values. Subtracting $xy + yz + zx = 0$ from each denominator we get

$$\begin{aligned}\frac{1}{x^2+2yz} + \frac{1}{y^2+2zx} + \frac{1}{z^2+2xy} &= \\ = \frac{1}{(x-y)(x-z)} + \frac{1}{(y-z)(y-x)} + \frac{1}{(z-x)(z-y)} &= \\ = \frac{(z-y)+(x-z)+(y-x)}{(x-y)(y-z)(z-x)} &= 0.\end{aligned}$$

This implies that among the original fractions (with absolute values), one of them is a sum of the other two. Hence the fractions don't fulfil triangle inequality and we reached the desired contradiction.

Answer. Possible values are all real numbers except for 0.

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3. *Let ABC be a triangle. The A-angle bisector intersects BC at D. Let E, F be the circumcenters of triangles ABD, ACD, respectively. Given that the circumcenter of triangle AEF lies on BC, find all possible values of $\angle BAC$.* (Patrik Bak)

Solution. Let O be the circumcenter of triangle AEF and denote $\alpha = \angle BAC$. Since $\angle BAD$ and $\angle CAD$ are acute (Fig. 2), points E, F lie in the half-plane BCA and the Inscribed angle theorem yields

$$\angle BED = 2 \cdot \angle BAD = \alpha = 2 \cdot \angle DAC = \angle DFC.$$

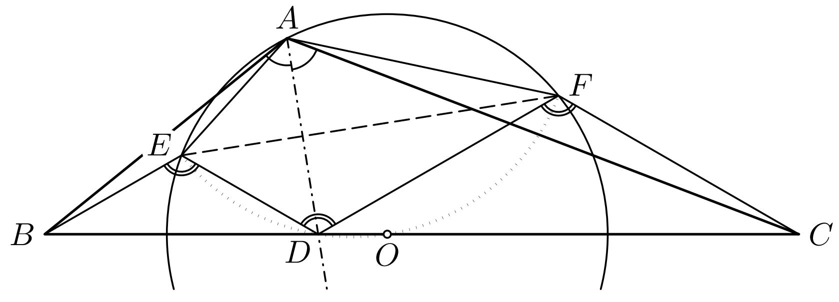

Fig. 2

The isosceles triangles BED and DFC are thus similar and we easily compute that $\angle EDF = \alpha$ and that BC is the external D-angle bisector in triangle DEF.

Point O lies on BC and on the perpendicular bisector of EF. Framed with respect to triangle DEF, it lies on the external D-angle bisector and on the perpendicular bisector of the opposite side EF. Thus it is the midpoint of arc EDF and $\angle EOF = \angle EDF = \alpha$.

Quadrilateral $AEDF$ is a kite, hence $\angle EAF = \alpha$. Moreover, line EF separates points A and O, thus the Inscribed angle theorem implies that the size of the reflex angle EOF is twice the size of the convex angle EAF. This yields $360° - \alpha = 2 \cdot \alpha$ and $\alpha = 120°$.

Answer. The only possible value is $\angle BAC = 120°$.

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4. *Consider positive integers a, b, c that are side-lengths of a non-degenerate triangle and such that* $\mathrm{GCD}(a, b, c) = 1$ *and the fractions*

$$\frac{a^2+b^2-c^2}{a+b-c}, \quad \frac{b^2+c^2-a^2}{b+c-a}, \quad \frac{c^2+a^2-b^2}{c+a-b}$$

are all integers. Prove that the product of the denominators of the three fractions is either a square or twice a square of an integer. (Jaromír Šimša)

Solution. Let $z = a+b-c$, $x = b+c-a$, $y = c+a-b$ be the (positive) denominators. Then $a = (y+z)/2$, $b = (x+z)/2$, $c = (x+y)/2$ and

$$a^2 + b^2 - c^2 = \tfrac{1}{4}((y+z)^2 + (x+z)^2 - (x+y)^2) = \tfrac{1}{2}(z(z+x+y) - xy),$$

hence $z \mid xy$ and likewise $y \mid xz$ and $x \mid yz$.

For a prime p, let i_p be the largest exponent such that $p^{i_p} \mid xyz$. It suffices to show that for all odd primes p the corresponding i_p is even. If i_2 is also even then xyz is a square. Otherwise, it is twice a square.

Fix odd prime p and consider the largest exponents α, β, γ such that $p^\alpha \mid x$, $p^\beta \mid y$, $p^\gamma \mid z$. Without loss of generality, assume $\min\{\alpha, \beta, \gamma\} = \gamma$. If $\gamma > 0$ then p divides each of x, y, z and thus it divides each of a, b, c (p is odd), contradicting $\mathrm{GCD}(a, b, c) = 1$. Therefore $\gamma = 0$.

From $x \mid yz$ we infer $\alpha \leqslant \beta$. Likewise, from $y \mid xz$ we infer $\beta \leqslant \alpha$. Hence $\beta = \alpha$ and $i_p = \alpha + \beta + \gamma = 2\alpha$ is an even number as desired.

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5. *Let $ABCD$ be an isosceles trapezoid with longer base AB. Let I be the incenter of triangle ABC and J the C-excenter of triangle ACD. Prove that IJ and AB are parallel.* (Patrik Bak)

Solution. Let K be the incenter of triangle ABD. Since $IK \parallel AB$, it suffices to show $JK \parallel AB$. Let $\angle ABD = \angle ACD = \phi$. Then $\angle AKD = 90° + \frac{1}{2}\phi$ and $\angle DJA = 90° - \frac{1}{2}\phi$, implying that the quadrilateral $AKDJ$ is cyclic (Fig. 3).

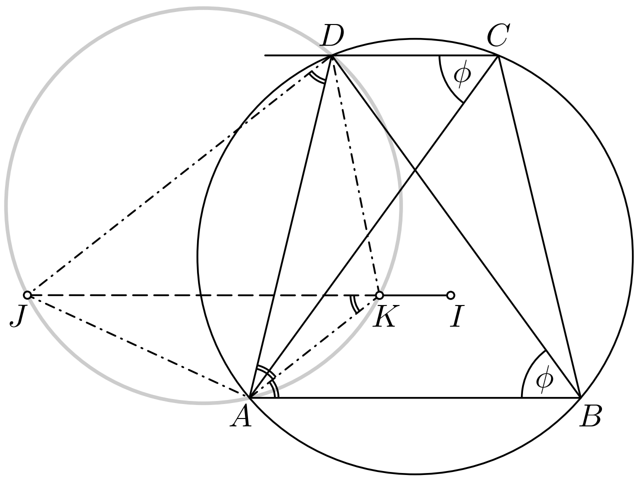

Fig. 3

As AK, DJ are bisectors of alternate interior angles, they are parallel. Together with the cyclic quadrilateral we obtain $\angle AKJ = \angle ADJ = \angle DAK = \angle KAB$ which concludes the proof.

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6. *Find the smallest positive integer n such that for any coloring of numbers $1, 2, 3, \ldots, n$ by three colors there exist two numbers with the same color whose difference is a square of an integer.* (Vojtech Bálint, Michal Rolínek, Josef Tkadlec)

Solution. The answer is $n = 29$.

First, for the sake of contradiction, assume that numbers $1, 2, \ldots, 29$ can be colored by colors A, B, C such that no two numbers with the same color differ by a square. Let $f(i)$ be the color of number i for $i \in \{1, 2, \ldots, 29\}$.

Since 9, 16, and 25 are squares, numbers 1, 10, 26 are all assigned distinct colors. The same is true for numbers 1, 17, 26, hence 10 and 17 are assigned the same color. Likewise we get $f(11) = f(18)$, $f(12) = f(19)$ a $f(13) = f(20)$ (for the last equality we look at numbers 4, 13, 20, 29).

Without loss of generality, assume $f(10) = f(17) = A$. As $11 = 10 + 1^2$, we have $f(11) \neq f(10)$. Without loss of generality, let $f(11) = f(18) = B$. Now $19 = 18 + 1^2 = 10 + 3^2$, hence $f(12) = f(19) = C$. Similarly, $20 = 19 + 1^2 = 11 + 3^2$ implies $f(13) = f(20) = A$. We have derived $f(13) = A = f(17)$, a contradiction.

On the other hand, if $n \leqslant 28$, we may color the numbers as below. It's easy to check that no two numbers with the same color differ by a square of an integer.

	1 B	2 C	3 A	4 C
5 A	6 B	7 C	8 B	9 C
10 A	11 B	12 C	13 B	14 C
15 A	16 B	17 A	18 B	19 C
20 A	21 B	22 A	23 B	24 C
25 A	26 C	27 A	28 B	

Fig. 4

Results of the Final Round

1.	Pavel Hudec	7	7	7	6	7	7	41
2.	Danil Koževnikov	6	7	7	7	7	6	40
3.	Matěj Doležálek	7	7	7	6	7	2	36
4.	Martin Raška	7	4	1	6	7	7	32
5.	Lenka Kopfová	7	3	1	5	7	7	30
6.	Josef Minařík	6	4	1	7	7	1	26
7.	Filip Čermák	7	3	1	7	7	0	25
8.	Radek Olšák	7	1	1	1	7	7	24
9.	Vít Jelínek	7	1	0	7	7	0	22
10.	Jonáš Havelka	7	3	0	4	1	7	22
11.	Filip Svoboda	5	3	0	6	7	0	21
12.	Jana Pallová	0	0	7	0	7	6	20
13.	Tomáš Perutka	7	0	1	4	7	0	19
14.	Tomáš Sourada	7	0	2	2	7	0	18
15.	Dalibor Kramář	7	3	0	0	7	0	17
16.	Václav Steinhauser	7	3	0	0	7	0	17
17.	Hedvika Ranošová	7	0	1	0	7	1	16
18.	Petr Gebauer	7	3	0	6	0	0	16
19.	Vít Pískovský	6	3	0	0	7	0	16
20.	Matěj Konvalinka	6	0	0	3	7	0	16
21.	Adam Janich	6	0	1	0	7	2	16
22.–23.	John Richard Ritter	7	0	0	0	7	0	14
	Martin Kurečka	4	0	0	4	6	0	14
24.–25.	Magdaléna Mišinová	2	0	0	4	7	0	13
	Václav Kubíček	7	3	0	1	0	2	13
26.	Adam Křivka	3	0	0	2	7	0	12
27.–29.	Jiří Vala	1	3	0	0	0	7	11
	Jindřich Jelínek	0	0	1	2	7	1	11
	Bára Tížková	1	0	1	0	7	2	11
30.–31.	Alexandr Jankov	1	0	1	2	6	0	10
	Tomáš Křižák	5	0	1	2	0	2	10
32.–35.	Matthew Dupraz	2	0	0	0	7	0	9
	Karel Chwistek	7	0	0	2	0	0	9
	Michal Košek	7	0	0	2	0	0	9
	Jiří Nábělek	0	0	1	0	4	4	9
36.	Martin Zimen	6	0	1	0	0	0	7
37.	Martin Schmied	1	3	0	2	0	0	6
38.	Petr Zahradník	2	3	0	0	0	0	5
39.–40.	Jiří Löffelmann	1	3	0	0	0	0	4
	Vojtěch David	1	1	0	2	0	0	4
41.–42.	Jan Hřebec	0	3	0	0	0	0	3
	Anna Mlezivová	1	1	0	0	1	0	3
43.	Daniela Opočenská	0	1	0	0	1	0	2