CAPS Match 2025 - solutions

ISTA, Austria

(First day – 17 June 2025)

1. Let a, b, c, d be nonnegative real numbers for which $a^2 + b^2 = ac + bd$ holds and c, d are not both zero. Find maximum and minimum value of the expression

$$\frac{ad + bc - cd}{c^2 + d^2}.$$

(*Michal Janík, Czech Republic*)

Solution 1. We will show that the maximum value is $\frac{1}{2}$ and the minimum is $-\frac{1}{2}$. For maximum, after some rearranging, we want to prove

$$2(ad + bc - cd) \leq c^2 + d^2,$$

or

$$2(ad + bc) \leq (c + d)^2.$$

After adding double the expression $ac+bd = a^2+b^2$ to both sides of this inequality, we will get equivalent inequality

$$2(a + b)(c + d) = 2(ad + bc + ac + bd) \leq (c + d)^2 + 2a^2 + 2b^2,$$

which can be further rearranged to

$$0 \leq (c + d)^2 - 2(a + b)(c + d) + (a + b)^2 + (a - b)^2 \leq (c + d - a - b)^2 + (a - b)^2,$$

which clearly holds. Moreover, this maximal value is reached by $a = b = c = d > 0$.

For the minimum value, notice that if $a \geq c$ or $b \geq d$ holds, $ad + bc \geq ad \geq cd$ and the expression is non-negative. So for it to be negative, both $a < c, b < d$ must hold and $a^2 + b^2 < ac + bd$ if a and b wouldn't both be 0. As they are equal by give condition, indeed $a = b = 0$ and he minimized expression then becomes $-\frac{cd}{c^2+d^2}$, which has minimum $-\frac{1}{2}$ as $2cd \leq c^2 + d^2$. Moreover, this minimal value is reached by $a = b = 0$ and $c = d > 0$

Solution 2. Consider the Cartesian coordinate system and in it, points $C = (c, 0), D = (0, d)$ and $X = (a, b)$. The line passing through C, D has equation $dx + cy - cd = 0$. From analytic geometry, the formula for distance of point (m, n) to the line $ix + jy + k = 0$ is known. It is $\frac{im+jn+k}{\sqrt{i^2+j^2}}$, where the distance is oriented according to the vertical position of point (m, n) with respect to the given line. With this formula, the distance of point X to the line through C, D is

$$\frac{ad + bc - cd}{\sqrt{c^2 + d^2}}.$$

By rearrranging the given condition on a, b, c, d, we get

$$\left(a - \frac{c}{2}\right)^2 + \left(b - \frac{d}{2}\right)^2 = \frac{c^2 + d^2}{4},$$

which means that the point $X = (a, b)$ lies on the circle with centre $\left(\frac{c}{2}, \frac{d}{2}\right)$ and radius $\frac{\sqrt{c^2+d^2}}{2}$, which is exactly the circle with diameter CD. Such point on circle with diameter CD can be at most radius distant from AB, so

$$\frac{ad + bc - cd}{\sqrt{c^2 + d^2}} \leq \frac{\sqrt{c^2 + d^2}}{2}$$

and by rearranging we get the inequality we proved in first solution. Note that the distance from X to CD is nonnegative unless $X = (0,0)$, as X would be above the line CD, because it lies in the first quadrant by the nonnegativity and the halfcircle with diameter CD in first quadrant lies entirely above the line CD. From this, the minimum value must happen for $X = (0,0)$ which gives $-\frac{1}{2}$, as desired.

2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that for every positive integer n

$$a_{n+1} = (n+1)\left(a_n - n + 1\right).$$

In terms of a_1, determine the greatest positive integer k such that $\gcd\left(a_i, a_{i+1}\right) = k$ for some positive integer $i \geq 2$. (Note that $\gcd(x,y)$ denotes the greatest common divisor of integers x and y.) (*Patrik Vrba, Slovakia*)

Solution. First, we will prove by induction that $a_n = (a_1 - 1)\, n! + n$ for all $n \geqslant 1$. The base case a_1 is trivial. Now suppose that the closed form holds for some a_n. Then

$$a_{n+1} = (n+1)\left(\left((a_1-1)\, n! + n\right) - n + 1\right)$$
$$a_{n+1} = (n+1)\left((a_1-1)\, n! + 1\right)$$
$$a_{n+1} = (a_1-1)\,(n+1)! + (n+1)$$

Hence, the proof by induction is complete. For the sake of clarity, let $c = a_1 - 1$. Let p be a prime number dividing $\gcd\left(a_n, a_{n+1}\right)$ for $n \geqslant 2$. Assume $p \leqslant n$. Then $cn! + n \equiv n \bmod p$ however, we have $c(n+1)! + n + 1 \equiv n + 1 \bmod p$ thus we conclude $p > n$. We have $a_{n+1} = (n+1)(cn! + 1)$ so then $p \mid (n+1)(cn!+1)$. Assume p divides $cn! + 1$. Then $cn! + n \equiv cn! + 1 \bmod p$, which implies $n \equiv 1 \bmod p$, which is a contradiction since $p > n$. So either $\gcd(a_n, a_{n+1}) = 1$ or $p = n + 1 = \gcd(a_n, a_{n+1})$.

Now suppose $p = n + 1$, notice that in this case $p \geq 3$. Then, according to Wilson's theorem $p \mid n! + 1$ so we have $p \mid n! + 1 \mid cn! + c$, which implies $p \mid cn! + c + n - n \Longrightarrow p \mid c - n \Longrightarrow p \mid c + 1$, which is however, equal to a_1. The chain of thoughts is reversible so we obtain $\gcd\left(a_{p-1}, a_p\right) = p$ if and only if $p \mid a_1$. Therefore, the answer is that k is the biggest odd prime divisor of a_1 or 1 if a_1 is a power of 2.

3. Maryam and Artur play a game on a board, taking turns. At the beginning, the polynomial $XY - 1$ is written on the board. Artur is the first to make a move. In each move, the player replaces the polynomial $P(X,Y)$ on the board with one of the following polynomials of their choice:

(a) $X \cdot P(X,Y)$

(b) $Y \cdot P(X,Y)$

(c) $P(X,Y) + a$, where $a \in (-\infty, 2025]$ is an arbitrary integer.

The game stops after both players have made 2025 moves. Let $Q(X,Y)$ be the polynomial on the board after the game ends. Maryam wins if the equation $Q(x,y) = 0$ has a finite and odd number of positive integer solutions (x,y). Prove that Maryam can always win the game, no matter how Artur plays. (*Daniel Holmes, Austria*)

Solution. We claim that Maryam can always achieve that the polynomial on the board at the end of her turn has the form $P(X,Y) = f(XY)$ where $f \in \mathbb{Z}[T]$ can be written as

$$T^n - \sum_{i=0}^{n-1} a_i T^i \qquad \text{for integers } n > 0 \text{ and } a_i \geq 0, \text{ not all of them zero.} \qquad (1)$$

As any such f fulfills $\frac{f(x)}{x^n} = 1 - \sum_{i=0}^{n-1} \frac{a_i}{x^{n-i}}$ for all positive real numbers x, which is a strictly increasing function on $(0, \infty)$ with arbitrarily small real values near 0 and tending to 1 for $x \to \infty$, it has exactly one positive real root r. We claim further that Maryam can choose r to be a perfect (integer) square. (Initially, $P(X,Y) = f(XY)$ with $f = T - 1$.) Maryam proceeds as follows:

- If Artur multiplies with X, Maryam multiplies with Y and if Artur multiplies with Y, Maryam multiplies with X. If initially, $P(X,Y) = f(XY)$ was on the board, then the resulting polynomial is $XYf(XY)$, so f changes to $T \cdot f$.

- If Artur adds an integer $0 \le a \le 2025$, Maryam adds the number $-a \le 2025$. The polynomial remains unchanged.

- If Artur adds an integer $a < 0$, write $A(XY)$ for the new polynomial on the board, where $A \in \mathbb{Z}[T]$ is of the form (1). By the discussion above, A has a unique positive real root u. As $A(x) > 0$ for all $x > u$, Maryam can choose an integer $c > u$ (e.g. $c = \lfloor u \rfloor + 1$) and add the negative integer $-A(c^2)$ to the polynomial A on the board. Then the new polynomial on the board has again the form (1) and (by construction) $c^2 \in \mathbb{Z}_{>0}$ as the only positive real root.

Hence, Maryam can always achieve that Q (the polynomial in the end of the game) satisfies $Q = g(XY)$ where g is of the form (1) and has a perfect square s^2, $s \in \mathbb{Z}_{>0}$, as unique positive real root. Now for all pairs of positive integers (x, y), we have $Q(x,y) = 0 \iff g(xy) = 0 \iff xy = s^2$ and it is well known that the number of solutions (x, y) to the last equation is (finite and) odd. (Pairs (x, y) and (y, x) with $x \neq y$ correspond and (s, s) is the only fixed point in this involution, giving an odd number overall.) Hence, Maryam can always win, independent of Artur's moves.

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(Second day – 18 June 2025)

4. The plane was divided by vertical and horizontal lines into unit squares. Determine whether it is possible to write integers into cells of this infinite grid so that:

(i) every cell contains exactly one integer

(ii) every integer appears exactly once

(iii) for every two cells A and B sharing exactly one vertex, if they contain integers a and b then at least one of the cells sharing a common side with both A and B contains an integer between a and b.

(*Marta Strzelecka and Michał Strzelecki, Poland*)

Solution. Yes, this is possible. Consider the spiral depicted below and write consecutive integers along the spiral:

−40	−39	−38	−37	−36	−35	−34	−33	−32
25	24	23	22	21	20	19	18	−31
26	−13	−12	−11	−10	−9	−8	17	−30
27	−14	5	4	3	2	−7	16	−29
28	−15	6	−1	0	1	−6	15	−28
29	−16	7	−2	−3	−4	−5	14	−27
30	−17	8	9	10	11	12	13	−26
31	−18	−19	−20	−21	−22	−23	−24	−25
32	33	34	35	36	37	38	39	40

We claim that this works. Consider any two cells A and B sharing exactly one vertex. Consider the 2×2 square containing A and B. If the 2×2 square contains a "corner" of the spiral then for some n and k the numbers in that 2×2 square are arranged in the following way (up to rotation or reflection):

$n+1$	k
n	$n-1$

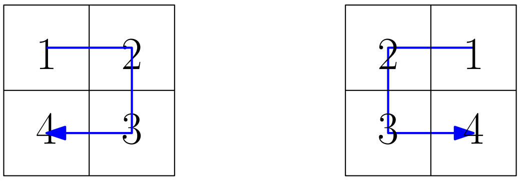
and therefore the conditions are satisfied no matter which opposite cells of the 2×2 square A and B are. Indeed, if A and B contain $n-1$ and $n+1$, then the good cell is the one containing n. If A and B contain n and k and $n < k$ then $n < n+1 < k$ and the good cell is the one containing $n+1$. If A and B contain n and k and $n > k$ then $k < n-1 < n$ and the good cell is the one containing $n-1$.

Otherwise, the numbers are arranged in the following way (again, up to rotation or reflection):

$k+1$	k
n	$n+1$

for some n, k, and again, the conditions are satisfied. Indeed, without loss of generality, assume $n < k$. Then $n < n+1 < k < k+1$. If A and B contain n and k then the good cell is the one containing $n+1$. Otherwise, A and B contain $n+1$ and $k+1$, and the good cell is the one containing k.

Alternatively, one can notice that condition (iii) from the problem statement means that we can orient each square 2×2 according to the increasing numbers as suggested in the picture below:

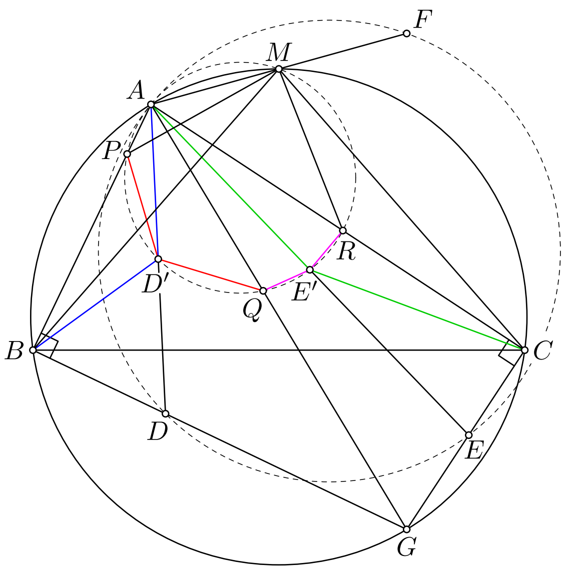


With this observation, it's relatively easy to check that the spiral construction satisfies this.

5. We are given an acute triangle ABC. Point D lies in the halfplane AB containing C and satisfies $DB \perp AB$ and $\angle ADB = 45^\circ + \frac{1}{2}\angle ACB$. Similarly, E lies in the halfplane AC containing B and satisfies $AC \perp EC$ and $\angle AEC = 45^\circ + \frac{1}{2}\angle ABC$. Let F be the reflection of A in the midpoint of arc BAC (containing point A). Prove that points A, D, E, F are concyclic. (*Patrik Bak, Slovakia*)

Solution 1. Denote $\angle ABC = \beta$ and $\angle ACB = \gamma$. The conditions translate as $\angle BAD = 45^\circ - \gamma$ and $\angle EAC = 45^\circ - \beta$. Denote by G the intersection point of BD and CE. Clearly $\angle BAG = 90^\circ - \gamma = 2\angle BAD$, and so AD is the angle bisector of BAG. Similarly, AE is the angle bisector of GAC.

Let D', E' be the midpoints of AD, AE, respectively. It is enough to show that the circle through A, D', E' also passes through the midpoint of arc BAC. Consider the circumcircle of $AD'E'$ and denote its second interesection points with AB, AG, AC by P, Q, R, respectively.



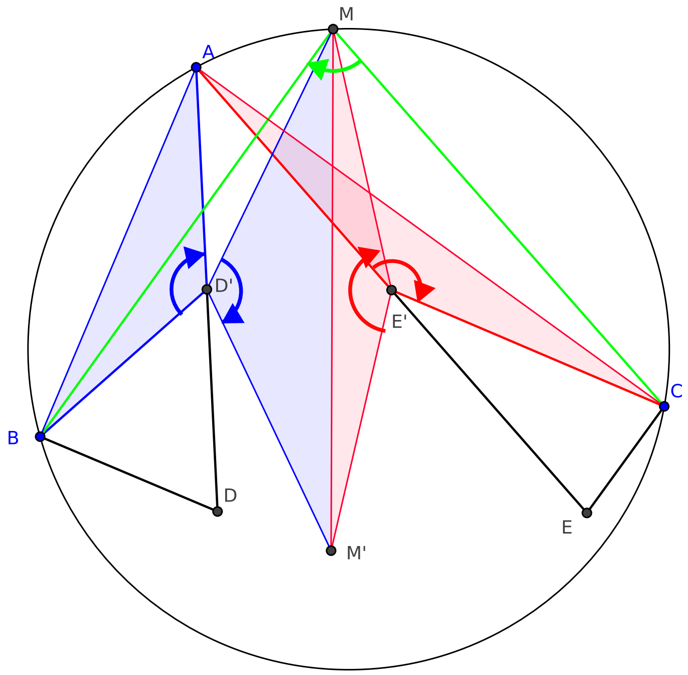
First, we will show that $BP = CR$. Notice that due $\angle ABD$ being right, we have that D' is the circumcenter of ABD, and so $D'B = D'A$. Then we get $\angle D'BA = \angle PAD' = \angle D'AQ$, and also $\angle AQD' = \angle BPD'$. Together with $D'B = D'A$, we have that triangles $D'AQ$ and $D'BP$ are congruent, and so $BP = AQ$. Similarly, we can show $CR = AQ$, and so $BP = CR$ as we wanted.

We will now show that the circle through A, P, R passes through the midpoint of arc BAC. Denote by M the second intersection of this circle with the circle ABC.

Clearly $\angle MBP = \angle MCR$ and $\angle MPA = \angle MRA$, and also $BP = RC$, so triangle MBP and MCR are congruent, giving $MB = MD$, which is enough.

Solution 2. Similarly to the previous solution, we consider homothety with center A and coefficient $1/2$ to obtain points D', E', M and prove that $AD'B$, $AE'C$, BMC are isosceles triangles. Moreover, by angle chasing we can get $\angle BMC = \alpha$, $\angle AD'B = 90° + \gamma$ and $\angle AE'C = 90° + \beta$. Let us notice that the sum of these angles $\angle BMC + \angle AD'B + \angle AE'C = 360°$. We may view these three isosceles triangles as three rotations (for example, triangle $AD'B$ corresponds to the rotation around D' by angle $AD'B$ and sends point B to point A). We will call them green, blue, and red.

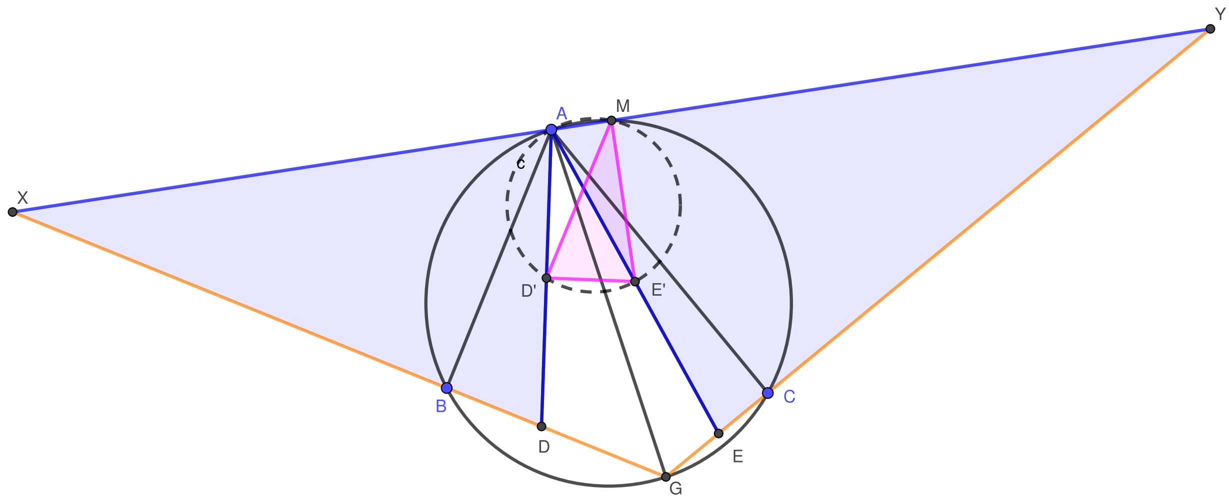
Because the sum of the three angles is 360°, the composition of these three rotations is a translation. Moreover, if we follow the image of C we notice that green rotation maps it to B, then blue maps it to A, and finally red maps it back to C. Hence, the translation is actually an identity. Let M' be the image of M under the blue rotation. Then $MD'M'$ is similar to $AD'B$. And because M is the center of the green rotation, the composition of blue and red rotations has to map M back to M. Hence, $M'E'M$ has to be similar to $AE'C$. And so $\angle D'ME' = \angle BAD' + \angle CAE' = \alpha/2 = \angle D'AE'$. Thus, $AME'D'$ is cyclic and we are done.



Solution 3. [sketch] Let G, D', E' be as in the original solution. Denote M the midpoint of arc BAC. Moreover, let the line AM meet the lines BD and EC at X and Y, respectively. Since AM is the external angle bisector, we get $\angle XAB = \angle CAY = 90° - \alpha/2$. Therefore, $\angle GXY = \angle GYX = \alpha/2$, so the triangle GXY is isosceles. Since AG is a diameter of the circumcircle of ABC, $GM \perp XY$, thus M is the midpoint XY.

We can calculate $\angle XAD = 180° - \alpha/2 - (45° + \gamma/2) = 45° + \beta/2$. Similarly $\angle CAY = 45° + \beta/2$. This gives us that the triangles DAX and ACY are spirally similar. From this spiral similarity we get that also $D'E'M$ is similar to them. So, $\angle D'ME' = \alpha/2$.

We can calculate $\angle D'AE' = 180° - (45° + \beta/2) - (45° + \gamma/2) = \alpha/2$, hence A, D', E', M are concyclic. Homothety with center A and coefficient 2 maps this circle to the desired circle.



6. Find all functions $f\colon (0,\infty) \to [0,\infty)$ such that for all $x, y \in (0,\infty)$ it holds that

$$f(x + yf(x)) = f(x)f(x + y).$$

(*Dominik Martin Rigász, Slovakia*)

Solution. Any f such that $f(x) \in \{0,1\}$ for all $x \in \mathbb{R}^+$works. Furthermore, any f such that

$$f(x) = \begin{cases} 0 \text{ or } 1 & x \in (0, x_0) \\ c & x = x_0 \\ 0 & x \in (x_0, \infty) \end{cases}$$

works as well, where $x_0 > 0, c \geq 0$ are arbitrary constants. We now show that these are the only solutions. For $f(x) \neq 0$ easy both-ways induction yields that for all $n \in \mathbb{Z}$ it is true that

$$f(x)^n f(x + y) = f\left(x + yf(x)^n\right) \tag{2}$$

Now assume there exist $0 < x_0 < x_1$ such that $f(x_0) \notin \{0,1\}$ and $f(x_1) \neq 0$ (if such a pair doesn't exist then f must have one of the two forms described above). Then substituting $[x_0, x_1 - x_0]$ into (1) and manipulating n (in particular we consider $n \to -\infty$ if $f(x_0) < 1$, and $n \to +\infty$ if $f(x_0) > 1$) yields that f reaches arbitrarily large values at arbitrarily large arguments. Hence, for every pair of positive reals c_1, c_2 there are infinitely many x such that $x > c_1$ and $f(x) > c_2$. Call this fact $(\star)$.

We now multiply the given equation by $f(x + y + z)$, where z is a positive real number, to get

$$f(x+y+z)f(x+yf(x)) = f(x)f(x+y)f(x+y+z) = f(x)f(x+y+zf(x+y)),$$

where we've used the property from the problem statement to obtain the second equality. We now choose z such that $z > yf(x) - y$. Then $x + y + z > x + yf(x)$. Hence, we can apply the problem statement on both the left-most side and the right-most side of the above equation to get

$$f(x+y+z)f(x+yf(x)) = f(x+yf(x)+(z-yf(x)+y)f(x+yf(x)))$$
$$f(x)f(x+y+zf(x+y)) = f(x+(y+zf(x+y))f(x))$$

Together with $f(x+yf(x)) = f(x)f(x+y)$, since the LHS's are equal in the above two equations, we get

$$f(x+yf(x)+(z-yf(x)+y)f(x)f(x+y))=f(x+(y+zf(x+y))f(x)). \quad (3)$$

If the arguments in the above equation were equal, then by simplification, this would yield the equivalent equality

$$(-yf(x)+y)f(x)f(x+y)=0. \quad (4)$$

We now choose x_0, y_0 such that $f\left(x_0\right) \notin\{0,1\}$ and $f\left(x_0+y_0\right) \neq 0$ and substitute $\left[x_0, y_0\right]$ into (2). Note that for this pair, equation (3) does not hold, and hence the arguments in (2) are always distinct. In particular, the arguments on both sides of (2) are linear functions in z with the same positive gradient (namely $f\left(x_0\right) f\left(x_0+y_0\right)$), but different y-intercept values. Since (2) holds for all large z (namely all $z>y_0 f\left(x_0\right)-y_0$), it follows that f is eventually periodic. Hence, there are constants $C, P>0$ (dependent on x_0, y_0), such that $f(x)=f(x+P)$ for all $x>C$.

By $(\star)$ we know that there is an $x_2>C$ such that $f\left(x_2\right) \notin\{0,1\}$. Then by comparing $\left[x_2, y\right]$ with $\left[x_2, y+P\right]$ in the original equation we get

$$f\left(x_2+y f\left(x_2\right)\right)=f\left(x_2+y f\left(x_2\right)+P f\left(x_2\right)\right),$$

since the RHS's remains the same (since $x_2+y>C$). Now let $y=\frac{P}{f\left(x_2\right)}$ in the above, to obtain

$$f\left(x_2+P\right)=f\left(x_2+P+P f\left(x_2\right)\right)$$

and hence $f\left(x_2\right)=f\left(x_2+P f\left(x_2\right)\right)=f\left(x_2\right) f\left(x_2+P\right)=f\left(x_2\right)^2$, clear contradiction.